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KEY

TO

HUTTON'S COURSE

OF

Mathematics.

Entered at Stationers' Hall.

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Black Horse Court, London.

38071

KEY

TO THE COURSE

OF

MATHEMATICS,

COMPOSED FOR

THE USE OF THE ROYAL MILITARY ACADEMY,

BY

CHARLES HUTTON, LL.D. F.R.S.

THE LATEST EDITION,
Enlarged and Corrected.

BY

DANIEL DOWLING,

PROFESSOR OF ASTRONOMY AND THE MATHEMATICS;
LECTURER ON NATURAL PHILOSOPHY;
AND MASTER OF THE CLASSICAL, COMMERCIAL, AND MILITARY SCHOOL,

Mansion-House, Highgate.

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1818.

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PREFACE.

HAVING several years ago introduced DR. HUTTON'S COURSE OF MATHEMATICS into the number of Class Books used in my own SEMINARY, it was convenience first prompted me to put in manuscript a KEY to the more useful parts of it.

After some time, and when Dr. Hutton had enlarged his work, I indeed substituted his Three Volumes, almost exclusively, for the excellent French Authors till then generally and deservedly in the hands of the Mathematical Pupils.

But though my manuscript was thus acquiring bulk, still I had no intention of submitting it to the Public, greatly as I was convinced of its value in the school: and it was only at the instigation of some friends, highly read in the Mathematics, that I was induced to complete the KEY.

I confess I had long been desirous of contributing to the advancement of a science I had sedulously cultivated from my earliest years; and, on mature deliberation, I thought I could in no way more effectually accomplish this object than by rendering Dr. Hutton's Treatise, if possible, of more extensive and general utility; being well aware that the want of a KEY to his comprehensive COURSE, had prevented many from deriving such essential benefit from it, as might otherwise have been expected.

Accordingly I took the opinion of several of the most eminent Mathematicians of the present day, who all, after perusal, so warmly extolled the solutions, and commended the plan, that I could be no longer in doubt what line to pursue.

It is on these grounds then I venture to lay my labours before an impartial public, in the hope they will be found useful, not only to every description of Students in the Mathematics, but also to Officers of the Army, Navy, and the Honourable Company's service, as well as to others who may wish to *renew* their acquaintance with so many elegant parts of Mathematical science as are to be found in the COURSE.

With regard to the arrangement of the KEY, it was necessary it should be the same as that pursued by Dr. Hutton in his Three Volumes; but the facility of finding any question or proposition is rendered greater than is often to be met with.

I have in general adhered, with scrupulous exactness, to the rules laid down for the Student; yet in some few instances I have deviated a little, where deviation was warrantable: and these cases more frequently occur, perhaps, in the solution of Examples by logarithms. Many theorems I have demonstrated in two or more ways, where it appeared to me necessary; and in *Analytical Plane*

Trigonometry there is given, with very few exceptions, an easy geometrical construction, as best calculated to be more generally serviceable.

It has likewise been deemed expedient, sometimes for illustration, and sometimes for the detection of error, to subjoin occasional notes and remarks. By these it will be seen, there are some imperfections in Dr. Hutton's text and Answers.

These imperfections, however, if the extent and multifariousness of the work be taken into consideration, are comparatively few; and, in alluding to them, I by no means wish to infer, that the KEY may be expected entirely free from error.

It has nevertheless been uniformly my aim throughout, to send it forth with as few blemishes as possible; and in this, I trust, I have succeeded nearly to my wishes. Should there, however, be discovered defects of which I am not at present aware, I shall highly esteem the favour of their being pointed out to me by letter, in order to correct them. I own *much* novelty is not to be looked for in a work of the present kind; though at the same time I think the reader will be gratified in some degree by often meeting it unexpectedly.

Here it may be allowed me to state, that I have spared no pains or expense to render the work beautiful in execution; and if the price at first sight be thought a little more than the Book might have been afforded for, the difficulty of printing such matter, and the scanty number of impressions in the edition, will argue that I could have had no mercenary object in view by fixing the price at £1 .. 4*s*. the copy.

The geometrical figures and *style* of printing, it is presumed, will be some enhancement to the value; while the size of the volume and the quality of the paper will enable and entitle it to be bound uniform with the COURSE to which it is the KEY.

But it is not for myself to judge how far I have succeeded; since, unless the book shall be useful to the public, my labours will in a measure be incomplete.

Mansion-House, Highgate,
November 15, 1817.

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ERRATA.

Page 165, last line, AC should be BC.
Page 172, last line, —64° should be $=64^\circ$.
Page 190, 21st line, 71° .. 16′ should be 37° .. 28′.
Page 261, 25th line, " solid" should be " the solid."
Page 271, 2nd line from the bottom, $\overline{32\frac{1}{6}}$ should be $\overline{32\frac{1}{6}}^2$.

KEY

TO

HUTTON's COURSE OF MATHEMATICS.

NUMERATION.

(Key to Vol. I. page 6.)

THIRTY-FOUR.
Ninety-six.
Three hundred and eighty.
Seven hundred and four.
Six thousand, one hundred and thirty-four.
Nine thousand and twenty-eight.
Fifteen thousand and eighty.
Seventy-two thousand and three.
One hundred and nine thousand, and twenty-six.
Four hundred and eighty-three thousand, five hundred.
Two millions, five hundred thousand, six hundred, and thirty-nine.
Seven millions, and five hundred and twenty-three thousand.
Thirteen mill. four hundred and five thous. six hundred and seventy.
Forty-seven mill. fifty thous. and twenty-three.
Three hundred and nine mill. twenty-five thous. six hundred.
Four thous. seven hund. and twenty-three mill. five hund. and seven thous. six hund. and eighty-nine.
Two hund. seventy-four thous. eight hund. and fifty-six mill. three hundred and ninety thousand.
Six billions, five hund. seventy-eight thous. six hund. millions, three hund. and seven thous. and twenty-four.

(Key to Vol. I. page 6.)

NOTATION.

57
286
9210
27594
640481
3260106
408255192
27008096204
200540110016
21000810064150

(Page 11.)

ADDITION.

Ex. 5.
3426
9024
5106
8890
1204
Ans. 27650

Ex. 6.
509267
235809
72920
8392
420
21
9
Ans. 826838

Ex. 7.
2
19
817
4298
50916
730205
9180634
Ans. 9966891

Qu. 8.		Quest. 9.		Quest. 10.	
Jan.	31	Apr.	16	52714	Inf.
Feb.	28	May	31	5110	Cav.
Mar.	31	Jun.	30	6250	Drag.
Apr.	30	July	31	3927	Lt. H.
May	31	Aug.	31	928	Art.
Jun.	30	Sep.	30	1410	Pion.
July	31	Oct.	31	250	Sapp.
Aug.	31	Nov.	24	406	Min.
Sep.	30	Ans.	224 Days.	Ans. 70995	Total.
Oct.	31				
Nov.	30				
Dec.	31				
Ans.	365 Days.				

(Key to Vol. I. page 12.)

SUBTRACTION.

Ex. 4.
From 5331806
Take 5073918
Rem. 257888
Proof 5331806

Ex. 5.
From 7020974
Take 2766809
Rem. 4254165
Proof 7020974

Ex. 6.
From 8503402
Take 574271
Rem. 7929131
Proof 8503402

Ques. 7. 1727
1642
Ans. 85 Years.

Ques. 8. 2548
1815
Ans. 733 Years.

Ques. 9. 4000
1656
Ans. 2344 Years.

Ques. 10. 1815
1150
Ans. 665 Years.

Ques. 11. 1441
1330
Ans. 111 Years.

Ques. 12. 1492
1302
Ans. 190 Years.

(Page 16.)

MULTIPLICATION.

Ex. 1.
Mul. 123456789
by 3
370370367 Prod.

Ex. 2.
Mul. 123456789
by 4
493827156 Prod.

Ex. 3.
Mul. 123456789
by 5
617283945 Prod.

Ex. 4.
Mul. 123456789
by 6
740740734 Prod.

(Key to Vol. I. page 16.)

Ex. 5. Mul. 123456789
by 7
864197523 Prod.

Ex. 6. Mul. 123456789
by 8
987654312 Prod.

Ex. 7. Mul. 123456789
by 9
1111111101 Prod.

Ex. 8. Mul. 123456789
by 11
1358024679 Prod.

Ex. 9. Mul. 123456789
by 12
1481481468 Product.

Ex. 10. Mul. 302914603
by 16
1817487618
302914603
4846633648 Prod.

Ex. 11. Mul. 273580961
by 23
820742883
547161922
6292362103 Prod.

Ex. 12. Mul. 402097316
by 195
2010486580
3618875844
402097316
78408976620 Prod.

Ex. 13. Mul. 82164973
by 3027
575154811
164329946
2464949190
248713373271 Prod.

Ex. 14. Mul. 7564900
by 579
68084100
52954300
37824500
4380077100 Prod.

(Key to Vol. I. page 16.)

Ex. 15. Mul. 8496427
by 874359

76467843
42482135
25489281
33985708
59474989
67971416

7428927415293 Prod.

Ex. 16. Mul. 2760325
by 37072

5520650
19322275
193222750
8280975

102330768400 Prod.

(Page 17.)

When there are Ciphers in the Factors.

Ex. 3. Multiply 81503600
by 7030

2445108000 Product by 30
5705252000 - - Product by 7000

572970308000 Product by 7030

Ex. 4. Multiply 9030100
by 2100

903010000 Product by 100
18060200 - - Product by 2000

18963210000 Product by 2100

Ex. 5. Multiply 8057069
by 70050

402853450 Product by 50
5639948300 - - Product by 70000

564397683450 Product by 70050

(Key to Vol. I. page 18.)

When the Multiplier is the Product of two or more Numbers in the Multiplication Table.

Ex. 2. Here the multiplier 36 is the product of 6 by 6; or, of 4 by 9; therefore (always prefering the product of two numbers to that of three or more) it is

31704592 *Multiplicand.*		31704592 *Multiplicand.*
6 First multiplier.		9 First multiplier.
190227552 *Prod. by* 6.	or,	285341328 *Prod. by* 9.
6 Second multiplier.		4 Second multipl.
1141365312 *Prod. by* $6 \times 6 = 36$.		1141365312 *Prod. by* $4 \times 9 = 36$.

Ex. 3. Here the multiplier 72 is the product of 12 by 6; or, of 8 by 9; therefore

29753804 *Multiplicand.*		29753804 *Multiplicand.*
12 First multiplier.		8 First multiplier.
357045648 *Prod. by* 12.	or,	238030432 *Prod. by* 8.
6 Second multipl.		9 Second multipl.
2142273888 *Prod. by* $6 \times 12 = 72$.		2142273888 *Prod. by* $9 \times 8 = 72$.

Ex. 4. The multiplier 96 being the product of 12 by 8, it follows that

7128368 *Multiplicand.*
by 12

85540416 *Prod. by* 12.
and by 8

684323328 *Prod. by* $8 \times 12 = 96$, which was required.

Ex. 5. In this example the multiplier 108 is the product of 12 by 9, therefore

160430800 *Multiplicand.*
12 First multiplier.

1925169600 *Prod. by* 12.
9 Second multiplier.

17326526400 *Prod. by* $9 \times 12 = 108$.

(Key to Vol. I. page 18.)

Ex. 6. Here the multiplier 1320 is the continued product of 10, 11, and 12; therefore,

$$\begin{array}{r l} 61835720 & \textit{Multiplicand.} \\ 11 & \text{First multiplier.} \\ \hline 680192920 & \textit{Prod. by } 11. \\ 10 & \text{Second multiplier.} \\ \hline 6801929200 & \textit{Prod. by } 10 \times 11 = 110. \\ 12 & \text{Third multiplier.} \\ \hline 81623150400 & \textit{Prod. by } 12 \times 10 \times 11 = 1320. \\ \hline \end{array}$$

Quest. 7. In this question the multiplier 500 is the continued product of 10 by 10 by 5, wherefore, $104 \times 10 \times 10 \times 5 = 104 \times 500$.

$$\begin{array}{r r l} & 104 & \textit{battalions.} \\ & 10 & \\ \cline{2-2} & 1040 & \textit{Prod. by } 10. \\ & 10 & \\ \cline{2-2} & 10400 & \textit{Prod. by } 10 \times 10 = 100. \\ & 5 & \\ \cline{2-2} \text{Ans.} & 52000 \text{ men.} & \textit{Prod. by } 5 \times 10 \times 10 = 500. \\ \cline{2-2} \end{array}$$

Or, more concisely, annex two ciphers to the multiplicand, and multiply by 5. Thus,

104, with two ciphers annexed, is 10400 *Prod. by* 100.
which $\times^{ed}$ by 5

gives the *Product by* 500 = 52000 men. Ans.

Quest. 8. Here, either 250 or 320 may be the multiplier; the former being the continued product of 5 by 5 by 10; and the latter of 4, 8, and 10. Therefore,

$$\begin{array}{r l c r l} 320 & \textit{waggons.} & & 250 & \textit{loaves.} \\ 5 & & & 4 & \\ \cline{1-1} \cline{4-4} 1600 & \textit{Prod. by } 5. & & 1000 & \textit{Prod. by } 4. \\ 5 & & & 8 & \\ \cline{1-1} \cline{4-4} 8000 & \textit{Prod. by } 25. & \text{or,} & 8000 & \textit{Prod. by } 32. \\ 10 & & & 10 & \\ \cline{1-1} \cline{4-4} \end{array}$$

Ans. 80000 loaves. *Prod. by* 250. Ans. 80000 loaves. *Prod. by* 320.

(Key to Vol. I. page 20.)

DIVISION.

Ex. 3.

```
4) 7 3 1 4 6 0 8 5 (18286521 1/4 Quotient.
   4 | | | | | | |
   ---
   33-' | | | | | |
   32   | | | | | |
   ---  | | | | | |
   11 --' | | | | |
    8     | | | | |
   ---    | | | | |
   34 ----' | | | |
   32       | | | |
   ---      | | | |
   26 -----' | | |
   24        | | |
   ---       | | |
   20 -------' | |
   20          | |
   ---         | |
   ·8 ---------' |
    8            |
   ---           |
   ·5 -----------'
    4
   ---
    1 remainder.
   ---
```

Ex. 4.

```
7) 53 1 7 9 8 6 0 2 7 (759712289 4/7 Quot.
   49 | | | | | | | |
   ---
   41-' | | | | | | |
   35   | | | | | | |
   ---  | | | | | | |
   67 --' | | | | | |
   63     | | | | | |
   ---    | | | | | |
   49 ----' | | | | |
   49       | | | | |
   ---      | | | | |
   ·8 -----' | | | |
    7        | | | |
   ---       | | | |
   16 -------' | | |
   14          | | |
   ---         | | |
   20 ---------' | |
   14            | |
   ---           | |
   62 -----------' |
   56              |
   ---             |
   67 -------------'
   63
   ---
    4 remainder.
   ---
```

(Key to Vol. I. page 20.)

Ex. 5. 12) 57 0 1 9 6 3 8 2 ($47516365\frac{2}{12}$ *Quotient.*

```
48
--
90-
84
--
61 --
60
--
19 ----
12
--
76 -----
72
--
43 ------
36
--
78 ---------
72
--
62 ----------
60
--
2 remainder.
--
```

Ex. 6. 37) 746 3 8 1 0 5 ($2017246\frac{3}{37}$ *Quotient.*

```
 74
 ---
 63-
 37
 ---
 268 --
 259
 ---
  91 ---
  74
 ---
 170 -----
 148
 ---
 225 -------
 222
 ---
   3 remainder.
 ---
```

(Key to Vol. I. page 20.)

Ex. 7.

97) 137 8 9 6 2 5 4 (1421610$\frac{84}{97}$ *Quotient.*
97
408
388
209
194
156
97
592
582
105
97
84 remainder.
97 divisor.

Ex. 8.

764)35821649(46886$\frac{745}{764}$ *Quot.*
3056
5261
4584
6776
6112
6644
6612
5329
4584
745 remainder.
764 divisor.

Ex. 9.

5201)72091365(13861$\frac{304}{5201}$ *Quo.*
5201
20081
15603
44783
41608
31756
31206
5505
5201
304 remainder.
5201 divisor.

(Key to Vol. I. page 20.)

Ex. 10. 57606)4637064283(80496$\frac{11707}{57606}$ *Quotient.*
460848
·285842
230424
554188
518454
357343
345636
·11707 remainder.
57606 divisor.

Quest. 11.
3)471(157 men. Ans.
3
17
15
21
21
· ·

Quest. 12.
18)378(21 miles. Ans.
36
18
18
· ·

Quest. 13. 365)£38330(£105$\frac{5}{365}$. Answer.*
365
1830
1825
· · · 5 remainder.
365 divisor.

---

* *This answer differs from that given in the Course.*

(Key to Vol. I. page 21.)

CONTRACTIONS IN DIVISION.

Ex. 2. 4)52619675 Ex. 3. 5)1379192

$13154918\frac{3}{4}$ *Quotient.* $275838\frac{2}{5}$ *Quotient.*

Ex. 4. 6)38672940 Dividend. Ex. 5. 7)81396627 Dividend.

6445490 *Quotient.* $11628089\frac{4}{7}$ *Quotient.*

Ex. 6. 8)23718920 Dividend. Ex. 7. 9)43981962 Dividend.

2964865 *Quotient.* $4886884\frac{6}{9}$ *Quotient.*

Ex. 8. 11)57614230 Dividend. Ex. 9. 12)27980373 Dividend.

$5237657\frac{3}{11}$ *Quotient.* $2331697\frac{9}{12}$ *Quotient.*

(Page 22.)

When Ciphers are annexed to the Divisor.

Ex. 3. 23000)7380964($320\frac{20964}{23000}$ *Quotient.**

· · · 69 · · ·

48

46

· 20964 remainder.

23000 divisor.

* *A small typographical error in the Course.*

Ex. 4. 5800)2304109($397\frac{1509}{5800}$ *Quotient.*

· · 174 · ·

564

522

421

406

· 1509 remainder.

5800 divisor.

(Key to Vol. I. page 23.)

When the Divisor is the Product of two or more Numbers in the Multiplication Table.

Ex. 2. Here the divisor 72 is the product of 8 by 9; or, of 12 by 6; therefore,

8)7014596 Dividend.

$$\left.\begin{array}{l}9)\ 876824\ldots\ldots\ldots\ldots 4 \text{ first remainder}\\ \quad\ \ 97424\ldots\ldots\ldots\ldots 8 \text{ second remainder}\end{array}\right\}=\frac{68}{72}$$

Hence $97424\frac{68}{72}$ *Quotient.*

or,

12)7014596 Dividend.

$$\left.\begin{array}{l}6)\ 584549\ldots\ldots\ldots\ldots 8 \text{ first remainder}\\ \quad\ \ 97424\ldots\ldots\ldots\ldots 5 \text{ second remainder}\end{array}\right\}=\frac{68}{72}$$

Hence, as before, $97424\frac{68}{72}$ *Quotient.*

Ex. 3. Here the divisor 132 is the product of 11 by 12; wherefore,

11)5130652 Dividend.

12) 466422...........10 first remainder.

38868.............6 second remainder.

Hence $38868\frac{76}{132}$ *Quotient.*

Ex. 4. In this example, 240, the divisor, is the continued product of 10, 2, and 12, that is, of 20 and 12; for which reason,

20)83016572 Dividend.

12) 4150828...........12 first remainder.

345902.............4 second remainder.

Whence it is evident that $345902\frac{92}{240}$ is the *quotient required.*

(Key to Vol. I. page 23.)

COMMON DIVISION, PERFORMED CONCISELY.

Ex. 2. 238)79165238(332627$\frac{12}{238}$ *Quotient.*
·776 ||||
·625 |||
1492 ||
··643 |
1678
··12 remainder.

Ex. 3. 5317)29137062(5479$\frac{5219}{5317}$ *Quotient.*
·25520 ||
·42526 |
·53072
·5219 remainder.

Ex. 4. 7803)62015735(7947$\frac{5294}{7803}$ *Quotient.*
·73947 ||
·37203 |
·59915
·5294 remainder.

(Page 31.)

REDUCTION.

Ex. 3. £24
20
480 shillings.
12
5760 pence.
4
Ans. 23040 farthings.

Ex. 4. 4)337587 farthings.
12) 84396$\frac{3}{4}$ pence.
20) 7033 shillings.
s. *d.*
Ans. £351 .. 13 .. 0$\frac{3}{4}$

(Key to Vol. I. page 31.)

Ex. 5. 36 guineas.
21
756 shillings.
12
9072 pence.
4
Ans. 36288 farthings.

Ex. 6. 4) 36288 farthings.
12) 9072 pence.
21) 756 shillings.
Ans. 36 guineas.

Ex. 7. lb. dwt. gr.
59 .. 13 .. 5
12
708 oz.
20
14173 dwt.
24
Ans. 340157 grains.

Ex. 8. 24)8012131 grains.
20) dwt. 333838 ..... 19 gr.
12) oz. 16691 ..... 18 dwt.
lb. 1390 ..... 11 oz.
Ans. lb. oz. dwt. gr.
1390 .. 11 .. 18 .. 19

Ex. 9. tons. cwt. qr. lb. oz. dr.
35 .. 17 .. 1 .. 23 .. 7 .. 13
20
717 cwt.
4
2869 qrs.
28
80355 lb.
16
1285687 oz.
16
Ans. 20571005 drams.

Qu. 10. 25000 miles.
8
200000 furlongs.
40
8000000 poles.
5½
44000000 yds.
3
132000000 feet.
12
1584000000 inches.
3
Ans. 4752000000 b. corns.

(Key to Vol. I. page 31.)

	days.	hrs.	min.	sec.
Quest. 11.	365 ..	5 ..	48 ..	$45\frac{1}{2}$
	24			
	8765 hours.			
	60			
	525948 minutes.			
	60			
Ans.	$31556925\frac{1}{2}$ seconds.			

	days.	hrs.	min.	sec.
Ques. 12.	29 ..	12 ..	44 ..	3
	24			
	708 hours.			
	60			
	42524 minutes.			
	60			
Ans.	2551443 seconds.			

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COMPOUND ADDITION.

		£	s.	d.
Ex. 2.	Ans.	59	18	$5\frac{1}{4}$
Ex. 3.	Ans.	79	7	1
Ex. 4.	Ans.	174	2	$4\frac{3}{4}$
Ex. 5.	Ans.	210	5	$11\frac{1}{4}$
Ex. 6.	Ans.	170	3	1
Ex. 7.	Ans.	141	11	$3\frac{1}{4}$
Ex. 8.	Ans.	930	4	$11\frac{1}{4}$

Quest. 9.	£	s.	d.	
	197	13	$7\frac{1}{2}$	Butcher's bill.
	59	5	$2\frac{3}{4}$	Baker's bill.
	85	0	0	Brewer's bill.
	103	13	0	Wine Merchant's bill.
	75	0	3	Corn-chandler's bill.
	27	15	$11\frac{1}{4}$	Cheese-monger's bill.
	55	3	$5\frac{3}{4}$	Tailor's bill.
	127	3	0	Sundry charges.
	100	0	0	Pocket-money.
	£830	14	$6\frac{1}{4}$	Sum Total.

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		lb.	oz.	dwt.
Ex. 1.	Ans.	235 ..	6 ..	15

		oz.	dwt.	gr.
Ex. 2.	Ans.	81 ..	3 ..	21

		℔	℥	ʒ	℈
Ex. 3.	Ans.	79 ..	9 ..	6 ..	1

		℥	ʒ	℈	gr.
Ex. 4.	Ans.	76 ..	2 ..	2 ..	15

(Key to vol. I. page 35.)

Ex. 5.	Ans.	lb. oz. dr. 70 .. 7 .. 7	Ex. 6.	Ans.	cwt. qr. lb. 54 .. 3 .. 22
Ex. 7.	Ans.	mil. fur. pol. 71 .. 1 .. 36	Ex. 8.	Ans.	yds. ft. in. 234 .. 0 .. 11
Ex. 9.	Ans.	yds. qr. nl. 331 .. 3 .. 1	Ex. 10.	Ans.	En.ells. qr. nl. 362 .. 1 .. 0
Ex. 11.	Ans.	A. R. P. 303 .. 3 .. 34	Ex. 12.	Ans.	A. R. P. 402 .. 3 .. 11
Ex. 13.	Ans.	tuns. hhd. gall. 137 .. 2 .. 51	Ex. 14.	Ans.	hhds. gall. pts. 86 .. 35 .. 0
Ex. 15.	Ans.	hhds. gall. pts. 56 .. 13 .. 6	Ex. 16.	Ans.	hhds. gall. pts. 124 .. 53 .. 7

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COMPOUND SUBTRACTION.

Ex. 3. Ans. £51 17 $7\frac{3}{4}$ Ex. 4. Ans. £217 2 $7\frac{1}{4}$

Quest. 5. £73 0 $5\frac{1}{4}$ minuend.
19 13 10 subtrahend.

Ans. £53 6 $7\frac{1}{4}$ remainder.

(Page 37.)

Quest. 6. £100 0 0 money lent by A.
73 12 $4\frac{3}{4}$ in value repaid by B.

Ans. £26 7 $7\frac{1}{4}$ balance in favour of A.

Quest. 7. Land-tax £0 14 6 } £20 12 0 half-year's rent.
Repairs 1 3 $3\frac{1}{4}$ } 1 17 $9\frac{1}{4}$ disbursed.

Ans. £18 14 $2\frac{3}{4}$ due.

(Key to Vol. I. page 37.)

Quest. 8.	Debts.			Effects.			
A.	£ 35	7	6	£23	7	5	Cash.
B.	91	13	0½	53	11	10¼	Wares.
C.	53	0	7¼	63	17	7¾	Household furn.
D.	87	5	0	25	7	5	Recovble debts.
E.	111	3	5¾				
				£166	4	4	

	£378	9	7½	amount of debts.
	166	4	4	value of effects.
Ans.	£ 212	5	3½	deficient.

EXAMPLES OF WEIGHTS AND MEASURES.

	lb. oz. dwt. gr.		lb. oz. dwt. gr.
Ex. 1.	Ans. 3 .. 10 .. 5 .. 17	Ex. 2.	Ans. 4 .. 2 .. 8 .. 5
	℔ ℥ ʒ ℈ gr.		cwt. qr. lb.
Ex. 3.	Ans. 43 .. 11 .. 2 .. 1 .. 15	Ex. 4.	Ans. 2 .. 1 .. 7
	lb. oz. dr.		mil. fur. pol.
Ex. 5.	Ans. 53 .. 11 .. 7	Ex. 6.	Ans. 6 .. 5 .. 6
	yds. ft. in.		yds. qr. nls.
Ex. 7.	Ans. 23 .. 0 .. 7	Ex. 8.	Ans. 8 .. 1 .. 3
	yd. qr. nl.		A. R. P.
Ex. 9.	Ans. 1 .. 2 .. 1	Ex. 10.	Ans. 0 .. 3 .. 6
			A. R. P.
		Ex. 11.	Ans. 34 .. 1 .. 27

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	tuns. hhds. gal.		hhds. gal. pts.
Ex. 12.	Ans. 8 .. 0 .. 50	Ex. 13.	Ans. 2 .. 50 .. 6
	hhds. gal. pts.		hhds. gal. pts.
Ex. 14.	Ans. 4 .. 47 .. 4	Ex. 15.	Ans. 52 .. 9 .. 4
	lasts. qr. bu.		B. G. P.
Ex. 16.	Ans. 3 .. 1 .. 2	Ex. 17.	Ans. 4 .. 4 .. 2
	mo. w. da.		da. hrs. min.
Ex. 18.	Ans. 54 .. 0 .. 6	Ex. 19.	Ans. 42 .. 6 .. 49

	yds. ft. in.	
Quest. 20.	236 .. 0 .. 0	line of defence.
	146 .. 1 .. 4	part terminated by the shoulder and curtain.
Ans.	89 .. 1 .. 8	length of the face required.

(Key to Vol. I. page 39.)

COMPOUND MULTIPLICATION.

Ex. 2. £0 7 8
4 lb.
£1 10 8 Ans.

Ex. 3. £0 0 9½
6 lb.
£0 4 9 Ans.

Ex. 4. £0 1 8½
7 lb.
£0 11 11½ Ans.

Ex. 5. £0 2 7½
9 st.
£1 3 7½ Ans.*

Ex. 6. £2 17 10
10 cwt.
£28 18 4 Ans.

Ex. 7. £3 7 4
12 cwt.
£40 8 0 Ans.

CONTRACTIONS.

When the Multiplier exceeds 12.

Ex. 2. £4 7 2
10 cwt.
£43 11 8 pr. of 10 cwt.
2 cwt.
Ans. £87 3 4 pr. of 20 cwt.

Ex. 3. £3 7 6
6 tons.
£20 5 0 pr. of 6 tons.
4 tons.
Ans. £81 0 0 pr. of 24 tons.

Ex. 4. £0 1 6
9 ells.
£0 13 6 price of 9 ells.
5 ells.
Ans. £3 7 6 price of 45 ells.

---

* Differing by the price of 1 stone of beef from the given Ans., £1. 1*s.* being the price of 8 stone.

(Key to Vol. I. page 40.)

Ex. 5.	£0	2	3	
			9	gallons.
	£1	0	3	price of 9 gals.
			7	gallons.
Ans.	£7	1	9	pr. of 63 gals.

Ex. 6.	£1	4	
		10	barrels.
	£12	0	pr. of 12 barrels.
		7	barrels.
Ans.	£84	0	pr. of 70 barrels.

Ex. 7.	£1	12	8	
			12	quarters.
	£19	12	0	price of 12 qrs.
			7	quarters.
Ans.	£137	4	0	pr. of 84 qrs.

Ex. 8.	£1	3	4	
			12	quarters.
	£14	0	0	price of 12 qr.
			8	quarters.
Ans.	£112	0	0	price of 96 qr.

Ex. 9.	£0	5	9	
			12	days.
	£3	9	0	wages for 12 d^{s}.
			10	days.
Ans.	£34	10	0	wages for 120 d^{s}.

Ex. 10.	£0	13	4	
			12	reams.
	£8	0	0	for 12 reams.
			12	reams.
Ans.	£96	0	0	for 144 reams.

* * *

When the Multiplier is not produced by the Multiplication of simple Numbers.

Ex. 2.	£2	5	3¼	
			4	quarters.
	£9	1	1	for 4 qrs.
			7	quarters.
	£63	7	7	for 28 qrs.
Add	£2	5	3¼	for 1 qr.
Ans.	£65	12	10¼	for 29 qrs.

Ex. 3.	£3	15	2	
			6	loads.
	£22	11	0	for 6 lds.
			9	loads.
	£202	19	0	for 54 lds.
Subtract	3	15	2	for 1 ld.
Ans.	£199	3	10	for 53 lds.

(Key to Vol. I. page 40.)

Ex. 4.	£0	11	$5\frac{3}{4}$	
			8	bushels.
	£4	11	10	for 8 bush.
			10	bushels.
	£45	18	4	for 80 bush.
Subtract		11	$5\frac{3}{4}$	for 1 bush.
Ans.	£45	6	$10\frac{1}{4}$	for 79 bush.

Ex. 5.	£0	12	2	
			12	casks.
	£7	6	0	for 12 casks.
			8	casks.
	£58	8	0	for 96 casks.
Add		12	2	for 1 cask.
Ans.	£59	0	2	for 97 casks.

Ex. 6.	£0	15	$3\frac{3}{4}$	
			12	stone.
	£9	3	9	for 12 stone.
			9	stone.
	£82	13	9	for 108 stone.
Add	£4	11	$10\frac{1}{2}$	for 6 stone.
Ans.	£87	5	$7\frac{1}{2}$	for 114 stone.

EXAMPLES OF WEIGHTS AND MEASURES.

lb. oz. dwt. gr.
Ex. 1. Ans. 143 .. 2 .. 12 .. 2

℔ ℥ ʒ ℈ gr.
Ex. 2. Ans. 20 .. 3 .. 6 .. 1 .. 0

cwt. qr. lb. oz.
Ex. 3. Ans. 355 .. 3 .. 6 .. 8

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mil. fur. pol. yds.
Ex. 4. Ans. 90 .. 7 .. 0 .. 2

yds. qrs. nls.
Ex. 5. Ans. 887 .. 2 .. 3

A. R. P.
Ex. 6. Ans. 260 .. 1 .. 3

tuns. hhds. gals. pts.
Ex. 7. Ans. 61 .. 3 .. 15 .. 6

wey. qr. b. p.
Ex. 8. Ans. 147 .. 1 .. 2 .. 2

mo. we. da. hr. mi.
Ex. 9. Ans. 1729 .. 2 .. 1 .. 0 .. 10

(Key to Vol. I. page 42.)

COMPOUND DIVISION.

Ex. 2. 3)£432 12 $1\frac{3}{4}$ Divd.
£144 4 $0\frac{1}{2}$ *Quot.*

Ex. 3. 4)£507 3 5 Divd.
£126 15 $10\frac{1}{4}$ *Quot.*

Ex. 4. 5)£632 7 $6\frac{1}{2}$ Divd.
£126 9 6 *Quot.*

Ex. 5. 6)£690 14 $3\frac{1}{4}$ Divd.
£115 2 $4\frac{1}{2}$ *Quot.*

Ex. 6. 7)£705 10 2 Divd.
£100 15 $8\frac{3}{4}$ *Quot.*

Ex. 7. 8)£760 5 6 Divd.
£ 95 0 $8\frac{1}{4}$ *Quot.*

Ex. 8. 9)£761 5 $7\frac{3}{4}$ Divd.
£ 84 11 $8\frac{3}{4}$ *Quot.*

Ex. 9. 10)£829 17 10 Divd.
£ 82 19 $9\frac{1}{4}$ *Quot.*

Ex. 10. 11)£937 8 $8\frac{3}{4}$ Dividend.
qr.
£ 85 4 5 $0\frac{7}{11}$ *Quotient.*

Ex. 11. 12)£1145 11 $4\frac{1}{4}$ Dividend.
qr.
£ 95 9 3 $1\frac{5}{12}$ *Quotient.*

When the Divisor exceeds 12, *and is the Product of two or more Numbers in the Multiplication Table.*

Ex. 2.
2)£150 6 8 for 20 cwt.
10)£ 75 3 4 for 10 cwt.
Ans. £ 7 10 4 for 1 cwt.

Ex. 3.
6)£ 98 8 Dividend.
6)£ 16 8 *Quotient* by 6.
Ans. £ 2 14 8 *Quot.* by 36.

(Key to Vol. I. page 42.)

Ex. 4.		Ex. 5.	
8)£71 13 10	Dividend.	8)£44 4 0	Dividend.
7)£ 8 19 $2\frac{3}{4}$	Quot. by 8.	12)£ 5 10 6	Quot. by 8.
Ans. £ 1 5 $7\frac{1}{4}$	Quot. by 56.	Ans. £ 0 9 $2\frac{1}{2}$	Quot. by 96.

Quest. 6. 4)£31 10 0 per cwt.
7)£ 7 17 6 per qr.
4)£ 1 2 6 per 4 lb.
Ans. £ 0 5 $7\frac{1}{2}$ per lb.

(Page 43.)

When the Divisor is not the Product of two or more Numbers in the Multiplication Table.

Ex. 2. 57)£39 14 $5\frac{1}{4}$ (£0 13 $11\frac{1}{4}$ *Quotient.*
20
) 794
224
·53
12
) 641
·71
14
4
) 57
· ·

(Key to Vol. I. page 45.)

```
Ex. 3.     43)£125   4`  9   (£2 18  3 Quotient.
               ·39
                20
               ----
              )784--
               354
              · 10
                12
               ----
              )129----
               ...
               ----

Ex. 4.     97)£542   7  10   (£5 11 10 Quotient.
               ·57
                20
               ----
             )1147--
             ·177
              ·80
               12
              ----
             )970-----
              ...
              ----

Ex. 5.    127)£123  11   2½  (£0 19  5½ Quotient.
                20
               ----
             )2471--
              1201
              ··58
                12
               ----
              )698----
               ·63
                 4
               ----
              )254------
               ...
               ----
```

EXAMPLES OF WEIGHTS AND MEASURES.

```
            lb.  oz. dwt. gr.                          ℔   ℥   ʒ   ℈   gr.
Ex. 1.   7)17 .. 9 .. 0 .. 2 Divd.    Ex. 2.   12)17 .. 5 .. 2 .. 1 .. 4 Div.
           ------------------                    --------------------
           2 .. 6 .. 8 .. 14 Quot.               1 .. 5 .. 3 .. 1 .. 12 Quo.
           ------------------                    --------------------
```

(Key to Vol. I. page 43.)

```
               cwt.  qr.  lb.   cwt. qr. lb.
Ex. 3.     53)178 .. 3 .. 14   (3 .. 1 .. 14 Quotient.
               ·19      |    |
                 4      |    |
              )79 -------    |
               26            |
               28            |
             )742 ------------
              212
              ...
```

```
         mil.  fur. po. yd. ft.  mil. fur. po. yds. ft.  in.
Ex. 4. 39)144 .. 4 .. 2 .. 1 .. 2  (3 .. 5 .. 25 .. 3 .. 2 .. 6 24/39 Quo.*
       |   27    |    |    |    |                   |
       |    8    |    |    |    |                   |
       | )220 ----    |    |    |                   |
       |  ·25         |    |    |                   |
       |   40         |    |    |                   |
       |)1002 ---------    |    |                   |
       | ·222              |    |                   |
       |  ·27              |    |                   |
       |   11              |    |                   |
       |2)299 half yds.-----    |                   |
       |   yds.  ft.  in.       |                   |
       |   149 .. 1 .. 6        |                   |
       |          2 .. 0 add -----                  |
       |-)150 .. 0 .. 6 (----------------------------
           33      |    |
           ·3      |    |
          )99 ------    |
           21           |
           12           |
         )258 -----------
          ·24 remainder.
           39 divisor.
```

---

* As this Quotient and Dr. Hutton's disagree, we subjoin the proof.

```
      mil. fur. po. yds. ft. in.
        3 .. 5 .. 25 .. 3 .. 2 .. 6 24/39
                                   39
                                           mil. fur. po. yd. ft. in.
Proof 144 .. 4 .. 2 .. 1½ .. 0 .. 6 evidently = 144 .. 4 .. 2 .. 1 .. 2 .. 0
```

(Key to Vol. I. page 43.)

```
                 yds.  qrs. nls. yds.  qr.  nls.
Ex. 5.        47)534 .. 2 .. 2 (11 .. 1 .. 2 Quotient.
                ·64     |    |
                 17     |    |
                  4     |    |
               ----     |    |
               )70 -----     |
                 23          |
                  4          |
               ----          |
               )94 ----------
                 ··
               ----
```

```
              A.    R.    P.   A.   R.    P.
Ex. 6.     51)71 .. 1 .. 33 (1 .. 1 .. 24 9/51 Quotient. *
              20    |     |
               4    |     |
            ----    |     |
            )81 ----      |
              30          |
              40          |
            ----          |
          )1233 ----------
           ·213
            ··9 remainder.
            ---
             51 divisor.
```

* If 77 acres be written in the Dividend instead of 71, the Quotient is 1 A. 2 R. 3 P., the Answer given in the Work.

```
              tuns. hhd. gal. pts. tu. hhd. gal. pts.
Ex. 7.     65)7 .. 0 .. 47 .. 7 (0 .. 0 .. 27 .. 7 Quotient.
              4     |     |     |
             --     |     |     |
             28 ----      |     |
             63           |     |
           ----           |     |
          )1811 ----------      |
           ·511                 |
            ·56                 |
              8                 |
           ----                 |
           )455 ----------------
            ···
           ----
```

(Key to Vol. I. page 43.)

$$\begin{array}{lrcccl}
 & \text{lasts.} & \text{qr.} & \text{bu.} & & \text{la. qr. bu.} \\
\text{Ex. 8.} & 72)387 & .. \; 9 & .. \; 0 & & (5 \; .. \; 3 \; .. \; 7 \;\; \textit{Quotient.} \\
 & \cdot 27 & & & & \\
 & 10 & & & & \\
 & \overline{)279} & & & & \\
 & \cdot 63 & & & & \\
 & 8 & & & & \\
 & \overline{)504} & & & & \\
 & \cdots & & & & \\
 & \overline{} & & & &
\end{array}$$

$$\begin{array}{lrcccl}
 & \text{mo.} & \text{we.} & \text{da.} & & \text{mo. we. da.} \\
\text{Ex. 9.} & 26)206 & \cdot\cdot \; 0 & \cdot\cdot \; 4 & & (7 \cdot\cdot 3 \cdot\cdot 5 \;\; \textit{Quotient.} \\
 & \cdot 24 & & & & \\
 & 4 & & & & \\
 & \overline{)96} & & & & \\
 & 18 & & & & \\
 & 7 & & & & \\
 & \overline{)130} & & & & \\
 & \cdots & & & & \\
 & \overline{} & & & &
\end{array}$$

RULE OF THREE.

(Page 46.)

Quest. 3. As $\overset{\text{yds.}}{9} : \overset{\text{yds.}}{72} \because 1 : 8, :: \overset{£.}{5} .. \overset{s.}{12} : (\overset{£.}{5} .. \overset{s.}{12}) \times 8 = £44 .. \overset{s.}{16}$. Ans.

Quest. 4. As $\overset{\text{days.}}{365} : \overset{\text{da.}}{1} :: £146 : £146 \div 365 = £0 .. \overset{s.}{8}$. Ans.

Quest. 5. As $\overset{\text{paces.}}{3} : \overset{\text{paces.}}{160} :: \overset{\text{yds.}}{2} : \frac{\overset{\text{yds.}}{160 \times 2}}{3} = \overset{\text{yds.}}{106} .. \overset{\text{ft.}}{2}$. Ans.

Quest. 6. As $\overset{\text{in.}}{9} : \overset{\text{in.}}{12}$, that is, As 3 : 4, :: $\overset{\text{in.}}{12} : \frac{\overset{\text{in}}{12 \times 4}}{3} = \overset{\text{in.}}{16}$. Ans.

Quest. 7. As $\overset{\text{men.}}{750} : \overset{\text{men.}}{1200}$, that is, As 5 : 8, :: $\overset{\text{rations.}}{22500} : \overset{\text{rations.}}{36000}$. Ans.

(Key to Vol. I. page 46.)

Quest. 8. $\text{As } \overset{\text{cwt.}}{7} .. \overset{\text{qr.}}{1} : \overset{\text{cwt.}}{43} .. \overset{\text{qr.}}{2} \because 1 : 6, :: \overset{\pounds.}{26} .. \overset{s.}{10} .. \overset{d.}{4} : \overset{\pounds.}{159} .. \overset{s.}{2}.$ Ans.

Quest. 9. $\text{As } \overset{\text{men.}}{750} : \overset{\text{men.}}{3500} \because 3 : 14, :: \overset{\pounds.}{2381} .. \overset{s.}{5} : \overset{\pounds.}{13212} .. \overset{s.}{10}.$ Ans.

Quest. 10. $\text{As } \overset{\text{ft.}}{3} \times \overset{\text{ft.}}{3} : \overset{\text{ft.}}{27} \times \overset{\text{ft.}}{20}, \text{ that is, As } 1 : 60, :: \overset{\text{yd.}}{1} : \overset{\text{yds.}}{60}.$ Ans.

Quest. 11. $\text{As } \overset{\text{bu.}}{36} : \overset{\text{bu.}}{6}, \text{ that is, As } 6 : 1, :: \overset{s.}{34} .. \overset{d.}{6} : \frac{\overset{s.}{34} .. \overset{d.}{6}}{6} = \overset{s.}{5} .. \overset{d.}{9}.$ Ans.

Quest. 12. $\text{As } \overset{\text{ft.}}{2} : \overset{\text{ft.}}{3} :: \overset{\text{stones.}}{6352} : \overset{\text{stones.}}{9528}.$ Ans.

Quest. 13. $\text{As } \overset{\text{oz.}}{1} : \overset{\text{lb.}}{73} .. \overset{\text{oz.}}{5} .. \overset{\text{dwt.}}{15} :: \overset{s.}{5} .. \overset{d.}{9} : \pounds 253 .. \overset{s.}{10} .. \overset{d.}{0\tfrac{3}{4}}.$ Ans.

Quest. 14. $\text{As } \overset{\text{men.}}{1124} : \overset{\text{men.}}{536} \because 281 : 134, :: \overset{\text{days.}}{365} : \overset{\text{days.}}{174\tfrac{16}{281}}.$ Ans.

Quest. 15. $\text{As } \pounds 1 : \overset{s.}{3} .. \overset{d.}{6} :: \pounds 763 .. \overset{s.}{15} : \pounds 133 .. \overset{s.}{13} .. \overset{d.}{1\tfrac{1}{2}}.$ Ans.

(Page 47.)

Quest. 16. $\text{As } \overset{\text{hrs.}}{6} : \overset{\text{hrs.}}{4}, \text{ that is, As } 3 : 2, :: \overset{\text{days.}}{12} : \overset{\text{days.}}{8}.$ Ans.

Quest. 17. $\text{As } \overset{\text{sh.}}{6} : \overset{\text{gs.}}{90} = \overset{\text{sh.}}{1890} \because 1 : 315, :: \overset{\text{bu.}}{1} : \overset{\text{bu.}}{315} = \overset{\text{qr.}}{39} .. \overset{\text{bu.}}{3}.$ Ans.

Quest. 18. $\text{As } \pounds 977 : \pounds 420 .. \overset{s.}{6} .. \overset{d.}{3\tfrac{1}{4}} :: \pounds 1 : \pounds 0 .. \overset{s.}{8} .. \overset{d.}{7\tfrac{1}{4}}.$ Ans.

Quest. 19. $\text{As } \overset{\text{horse.}}{2000} : \overset{\text{horse.}}{3000}, \text{ that is, As } 2 : 3, :: \overset{\text{days.}}{18} : \overset{\text{days.}}{27}.$ Ans.

Quest. 20. $\text{As } \overset{\text{day.}}{1} : \overset{\text{days.}}{365} :: \pounds 1 .. \overset{s.}{5} .. \overset{d.}{6} : \pounds 465 .. \overset{s.}{7} .. \overset{d.}{6}.$

$\text{But } \pounds 630 - \pounds 465 .. \overset{s.}{7} .. \overset{d.}{6} = \pounds 164 .. \overset{s.}{12} .. \overset{d.}{6}.$ Ans.

Quest. 21. $\text{As } \overset{\text{cwt.}}{1} : \overset{\text{cwt.}}{33} .. \overset{\text{qr.}}{0} .. \overset{\text{lb.}}{24} :: \pounds 0 .. \overset{s.}{16} .. \overset{d.}{4} : \pounds 27 .. \overset{s.}{2} .. \overset{d.}{6}.$ Ans.

Quest. 22. $\text{As } \overset{\text{da.}}{80} : \overset{\text{da.}}{54} :: \overset{\text{lb.}}{1\tfrac{1}{2}} : \overset{\text{lb.}}{1\tfrac{1}{80}}.$ Ans.

(Key to Vol. I. page 47.)

s. d. week. weeks.
Quest. 23. As 10 .. 6 : £20 : : 1 : 38$\frac{12}{126}$. Ans.

ch. ch. bu. s. d. s. d.
Quest. 24. As 1 : 75 .. 7 : : £1 .. 13 .. 6 : £125 .. 19 .. 0$\frac{1}{6}$. Ans.

s. d. s. d. oz. oz. dr.
Quest. 25. As 8 .. 4 : 7 .. 3 : : 8 : 6 .. 15$\frac{36}{100}$. Ans.

A. A. R. P. s. d. s. d.
Quest. 26. As 1 : 173 .. 2 .. 14 : : £1 .. 7 .. 8 : £240 .. 2 .. 7$\frac{1}{20}$. Ans.

cwt. cwt. qr. s. s. d. qr.
Qu. 27. As 19$\frac{1}{2}$: 132 .. 1$\frac{1}{4}$: : £10 .. 4 : £69 .. 4 .. 2 .. 1$\frac{13}{78}$. Ans.

qrs. yds. qrs. qrs. yds. yds. qrs. nls.
Quest. 28. As 3 : 3$\frac{1}{2}$=14 : : 7=1$\frac{3}{4}$: 8 .. 0 .. 2$\frac{2}{3}$. Ans.

yds. yds. qr. s. d. s. d.
Quest. 29. As 5 : 191 .. 1 : : 14 .. 2 : £27 .. 1 .. 10$\frac{1}{2}$. Ans.

days. day. s. s. d.
Quest. 30. As 365 : 1 : : £2107 .. 12 − £500 : £4 .. 8 .. 1$\frac{19}{365}$. Ans.

(Page 48.)

po. po. po. po. yds. ft. in.
Quest. 31. As 13$\frac{1}{2}$: 40 : : 4 : 11 .. 4 .. 2 .. 0$\frac{18}{27}$. Ans.

qrs. qrs. s. d. s. d.
Quest. 32. As 4 : 266 ∴ 2 : 133, : : 7 .. 9$\frac{1}{2}$: £25 .. 18 .. 1$\frac{3}{4}$. Ans.

cwt. qr. cwt. qr. lb. mil. mil. fur. po.
Quest. 33. As 3 .. 1 : 5 .. 0 .. 14 : : 96 : 151 .. 3 .. 3$\frac{1}{13}$. Ans.

oz. lb. oz. dwt. s. d. s. d.
Quest. 34. As 1 : 1 .. 7 .. 14 : : 6 .. 4 : £6 .. 4 .. 9$\frac{1}{5}$. Ans.

s. d.
A. A. s. d. s. d.
Quest. 35. As 1 : 547 : : $\frac{15..9}{2}$=7 .. 10$\frac{1}{2}$: £211 .. 19 .. 3. Ans.

men. men.
Quest. 36.* As $\frac{9\text{ ft.}}{6\text{ da.}}$: $\frac{27\text{ ft.}}{4\text{ da.}}$ ∴ 6 : 27 ∴ 2 : 9, : : 16 : 72. Ans.

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* This is a question in Compound Proportion. See Hutton's Practical Arithmetic, Quest. 5, page 38. *Lond.* 1815.

(Key to Vol. I. page 48.)

Quest. 37. As $\overset{\text{day}}{1} : \overset{\text{days.}}{365} :: \overset{d.}{14\frac{1}{2}} \times 20 = \pounds 1 .. \overset{s.}{4} .. \overset{d.}{2} : \pounds 441 .. \overset{s.}{0} .. \overset{d.}{10}$. Ans.

Quest. 38. As $\overset{\text{ells.}}{18} \times \overset{\text{qrs.}}{3} : \overset{\text{ells.}}{50} \times \overset{\text{qrs.}}{4} :: \pounds 1 .. \overset{s.}{19} .. \overset{d.}{6} : \pounds 7 .. \overset{s.}{6} .. \overset{d.}{3\frac{15}{27}}$. Ans.

Quest. 39. As $\overset{\text{yd.}}{1} \times \overset{\text{ft.}}{2\frac{1}{2}} : \overset{\text{yds.}}{20} \times \overset{\text{ft.}}{9} = 180 :: \overset{\text{yd.}}{1} : \overset{\text{yds.}}{72}$. Ans.

Quest. 40. As $\pounds 1 : \pounds 384 .. \overset{s.}{16} :: \pounds 0 .. \overset{s.}{2} .. \overset{d.}{9\frac{1}{2}} : \pounds 53 .. \overset{s.}{14} .. \overset{d.}{2\frac{1}{2}}$.

But $\pounds 384 .. \overset{s.}{16} - \pounds 53 .. \overset{s.}{14} .. \overset{d.}{2\frac{1}{2}} = \pounds 331 .. \overset{s.}{1} .. \overset{d.}{9\frac{1}{2}}$. Ans.

Quest. 41. As $\overset{\text{hours.}}{23} .. \overset{\text{min.}}{56} : \overset{\text{hour.}}{1} :: \overset{\text{miles.}}{25000} : \overset{\text{miles.}}{1044\frac{816}{1436}}$. Ans.

Quest. 42. As $\overset{s.}{10} : \overset{s.}{8}$, that is, As $5 : 4, :: \overset{\text{bot.}}{20} : \overset{\text{bottles.}}{16}$. Ans.

Quest. 43. As $\overset{\text{qr.}}{1} : \overset{\text{qr.}}{43} .. \overset{\text{bu.}}{5} :: \pounds 1 .. \overset{s.}{8} .. \overset{d.}{6} : \pounds 62 .. \overset{s.}{3} .. \overset{d.}{3\frac{3}{4}}$. Ans.

Quest. 44. As $\overset{\text{qrs.}}{5} \times \overset{\text{yd.}}{1} : \overset{\text{yd.}}{1} :: \overset{\text{qrs.}}{3} \times \overset{\text{yds.}}{50} = 150 : \overset{\text{yds.}}{30}$. Ans.

Quest. 45. As $\overset{\text{oz.}}{1} : \overset{\text{gr.}}{1}$, that is, As $480 : 1, :: \pounds 4 .. \overset{s.}{4} : \pounds 0 .. \overset{s.}{0} .. \overset{d.}{2\frac{1}{10}}$. Ans.

Quest. 46. As $\overset{\text{cwt.}}{3} : \overset{\text{lb.}}{1} :: \overset{\pounds.}{40} .. \overset{s.}{12} + \overset{\pounds.}{10} = \overset{\pounds.}{50} .. \overset{s.}{12} : \pounds 0 .. \overset{s.}{3\frac{4}{336}}$. Ans.*

* There is a minute error in the Denominator of the Fraction given in the last Edition of the Course, although in the Sixth Edition the Answer is right.

COMPOUND PROPORTION.

(Page 50.)

Quest. 2. As $\pounds 100 \times \overset{\text{yd.}}{1} : \pounds 750 \times \overset{\text{yds.}}{7} \because 2 : 105 :: \pounds 5 : \pounds 262 .. \overset{s.}{10}$. Ans.

Quest. 3. As $\overset{\text{pers.}}{8} \times \overset{\text{mo.}}{9} : \overset{\text{pers.}}{18} \times \overset{\text{mo.}}{12} \because 1 : 3, :: \pounds 200 : \pounds 600.$† Ans.

† Dr. Hutton's Answer to this Question is half the true Answer.

(Key to Vol. I. page 50.)

Quest. 4. As $\overset{\text{men}}{4} \times \overset{\text{da.}}{7} : \overset{\text{men}}{14} \times \overset{\text{da.}}{10} \because 1 : 5, :: £1..\overset{s.}{7} : £6..\overset{s.}{15}$. Ans.

Quest. 5. As $\overset{\text{mil.}}{130} \times \overset{\text{hrs.}}{10} : \overset{\text{mil.}}{360} \times \overset{\text{hrs.}}{12} \because 65 : 216, :: \overset{\text{days}}{3} : \overset{\text{days}}{9\tfrac{63}{65}}$. Ans.

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Quest. 6. As $\overset{\text{bu.}}{120} \times \overset{\text{hor.}}{6} : \overset{\text{bu.}}{94} \times \overset{\text{hor.}}{14} :: \overset{\text{days}}{56} : \overset{\text{days}}{102\tfrac{16}{45}}$. Ans.

Quest. 7. As $\overset{\text{men}}{340} \times \overset{\text{da.}}{15} : \overset{\text{men}}{120} \times \overset{\text{da.}}{25} :: \overset{\text{lb.}}{3000} : \overset{\text{lb.}}{1764}..\overset{\text{oz.}}{11\tfrac{15}{51}}$. Ans.

Quest. 8. As $\overset{\text{da.}}{12} \times \overset{\text{pers.}}{8} : \overset{\text{da.}}{365} \times \overset{\text{pers.}}{16}$, that is, As $6 : 365, :: \overset{\text{bar.}}{1} : \overset{\text{barrels}}{60\tfrac{5}{6}}$. Ans.

Quest. 9. As $\dfrac{\overset{\text{yds.}}{200} \times \overset{\text{ft.wd.}}{3} \times \overset{\text{ft.dp.}}{2}}{180\text{men} \times 10\text{hrs.}} : \dfrac{\overset{\text{yds.}}{360} \times \overset{\text{ft.wd.}}{4} \times \overset{\text{ft.dp.}}{3}}{100\text{men} \times 8\text{ hrs.}} :: \overset{\text{days}}{6} : \overset{\text{days}}{48\tfrac{6}{10}}$. Ans.*

* The Answer to this Question, as given in the "*Course of Mathematics*," is less than a third part of the Answer arising from the Proportion.

NOTE. *It has been thought proper, not only in the Rule of Proportion, but in other parts of the* KEY, *to set words over the figures, specifying the objects to which they allude; yet it must be distinctly understood that the numbers are every where* (square and solid measure excepted) *to be considered as taken abstractedly.*

REDUCTION OF VULGAR FRACTIONS.

(Page 53.)

```
Quest. 2.      246)372(1
                   126)246(1
                       120)126(1
The last divisor is - - - -  · · 6)120(20
                                   ...
                                   ───
          Therefore 6.   Ans.
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(Key to Vol. I. page 53.)

Quest. 3. 324)612(1
288)324(1
Last divisor - - - - - 36)288(8
...

And, 36)1032(28
·312
·24)36(1
Last divisor - - - - 12)24(2
..

Therefore 12. Ans.

(Page 55.)

To abbreviate Fractions.

Ex. 2. $195\left\{\frac{195}{780}\right\}=\frac{1}{4}$ Lowest terms.

Ex. 3. $68\left\{\frac{136}{204}\right\}=\frac{2}{3}$ Lowest terms.

Ex. 4. $105\left\{\frac{525}{630}\right\}=\frac{5}{6}$ Lowest terms.

To reduce a Mixed Number to its Equivalent Improper Fraction.

Ex. 2. $\frac{12\times9+7}{9}=\frac{115}{9}$ fraction required.

Ex. 3. $\frac{14\times10+7}{10}=\frac{147}{10}$ fraction required.

Ex. 4. $\frac{183\times21+5}{21}=\frac{3848}{21}$ fraction required.

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To reduce an Improper Fraction to an Equivalent Whole or Mixed Number.

Ex. 4. $56\div7=8$. equivalent integer.

(Key to Vol. I. page 56.)

Ex. 5. $1362 \div 25 = 54\frac{12}{25}$. equivalent mixed number.

Ex. 6. $2918 \div 17 = 171\frac{11}{17}$. equivalent mixed number.

(Page 57.)

To reduce a whole Number to an equivalent Fraction with a given Denominator.

Ex. 2. $\frac{12 \times 13}{13} = \frac{156}{13}$ fraction required.

Ex. 3. $\frac{27 \times 11}{11} = \frac{297}{11}$ fraction required.

(Page 58.)

To reduce a Compound Fraction to an equivalent Simple Fraction.

Ex. 3. $3 \times 4 = 12$ numerator,
and $7 \times 5 = 35$ denominator.
Therefore $\frac{12}{35}$ fraction required.

Ex. 4. $2 \times 3 \times 5 = 30$ numerator,
and $3 \times 5 \times 9 = 135$ denominator.
Hence $\frac{30}{135} = \frac{2}{9}$ Ans.

Or,

Cancelling 3 and 5 in the Numerator and also in the Denominator of the compound fraction, the simple fraction is $\frac{2}{9}$ *as before.*

Ex. 5. $3\frac{1}{2} = \frac{7}{2}$, And $\frac{\cancel{2} \times \cancel{5} \times 7}{\cancel{5} \times 8 \times \cancel{2}} = \frac{7}{8}$ fraction required.

Ex. 6. $\frac{\cancel{2} \times 5 \times \cancel{7} \times \cancel{4}}{\cancel{7} \times \underset{2}{\cancel{8}} \times \cancel{2} \times 1} = \frac{5}{2}$ fraction required.

Ex. 7. $\frac{2 \times 5}{5 \times 6} = \frac{1}{3}$, and $2\frac{1}{3} = \frac{7}{3}$ fraction required.

(Key to Vol. I. page 58.)

To reduce Fractions of different Denominators to equivalent Fractions having a common Denominator.

Ex. 2. $2\times9=18$ First numr.
$5\times7=35$ Sec. numr. Therefore $\frac{18}{63}$ and $\frac{35}{63}$ frac^{s}. required.
And $7\times9=63$ Com. denr.

(Page 59.)

Ex. 3. $2\times5\times4=40$ First numr.
$3\times3\times4=36$ Sec. numr.
$3\times3\times5=45$ Third numr.
And $3\times5\times4=60$ Com. denr.
Hence $\frac{40}{60}$, $\frac{36}{60}$, and $\frac{45}{60}$ Ans.

Ex. 4. $2\frac{3}{5}=\frac{13}{5}$, and $4=\frac{4}{1}$.

Then $5\times5\times1=25$ First numerator.
$13\times6\times1=78$ Second numerator.
$4\times5\times6=120$ Third numerator.
And $6\times5\times1=30$ Common denominator.

Wherefore $\frac{25}{30}$, $\frac{78}{30}$, and $\frac{120}{30}$ are the fractions required.

(Page 60.)

To find the Value of a Fraction in parts of the Integer.

Ex. 3. 3 numerator.
20 next inferior denomination.
s. *d.*
denr. 8) 60 (7 .. 6 value sought.

Ex. 4. 2 numerator.
21 next inferior denomination.

denr. 9) 42 Product.

4*s.* 8*d.* Ans.

(Key to Vol. I. page 60.)

Quest. 5.
s. *d.*
2 .. 6
3 numerator of the given fraction.
denr. 4) 7 .. 6 Product.
s. *d.*
1 .. $10\frac{1}{2}$ Ans.

Quest. 6.
s. *d.*
4 .. 10
2 numerator of the given fraction.
denr. 5) 9 .. 8 Product.
s. *d.*
1 .. $11\frac{1}{5}$ Ans.

Quest. 7.
4 lb. Troy. numr. of the given fraction.
12 next inferior denomination.
denr. 5) 48 ounces.
dwt.
oz. 9 .. 12. Ans.

Quest. 8.
5 numerator of the given fraction.
4 next inferior denomination.
denr. 16) 20 quarters.
qr. lb.
1 .. 7 Ans.

Quest. 9.
7 numerator of the given fraction.
4 next inferior denomination.
denr. 8) 28 roods.
ro. poles.
3 .. 20. Ans.

(Key to Vol. I. page 60.)

Quest. 10. 3 numerator of the given fraction.
24 next inferior denomination.

denr. 10) 72 hours.

hrs. min.
7 .. 12 Ans.

(Page 61.)

To reduce a Fraction from one Denomination to another.

Ex. 3. $\pounds\frac{2\times\overset{s.}{20}\times\overset{d.}{12}}{15\times1\times1}=\frac{\overset{d.}{480}}{15}=\frac{32}{1}d.$ value sought.

Ex. 4. q. $\frac{2}{5}\times\frac{1}{4}\times\frac{1}{12}\times\frac{1}{20}=\pounds\frac{1}{2400}$ fraction required.

Ex. 5. Cwt. $\frac{2\times\overset{qr.}{4}\times\overset{lb.}{28}}{7\times1\times1}=$ lb. $\frac{32}{1}$ fraction required.

Ex. 6. Dwt. $\frac{3}{5}\times\frac{1}{20}\times\frac{1}{12}=\frac{1}{400}$ lb. Troy.

Ex. 7. Cr. $\frac{3}{8}\times\frac{5}{1}\times\frac{1}{21}=\frac{5}{56}$ of a guinea.

Ex. 8. H. cr. $\frac{5}{6}\times\frac{5}{2}=\frac{25}{12}$ of a shilling.

Ex. 9. $\overset{s.}{2}$ .. $\overset{d.}{6}=\overset{d.}{30}$, and $\pounds1=\overset{d.}{240}$, therefore
$\pounds\frac{30}{240}=\pounds\frac{1}{8}$ is the fraction required.

(Key to Vol. I. page 61.)

Ex. 10. s. d. qr.
17 .. 7 .. $3\frac{3}{5}$ numerator.
12

211 pence.
4

847 farthings.
5

Numerator 4238 fifth-parts of a farthing.

£1 denominator.
20

20 shillings.
12

240 pence.
4

960 farthings.
5

Denominator 4800 fifth-parts of a farthing.

Wherefore $£\frac{4238}{4800} = £\frac{2119}{2400}$ is the fraction required.

ADDITION OF VULGAR FRACTIONS.

(Page 62.)

Ex. 4. $\frac{3+6}{7} = \frac{9}{7} = 1\frac{2}{7}$. Ans.

Ex. 5. $\frac{3}{4} = \frac{27}{36}$, and $\frac{5}{9} = \frac{20}{36}$. But $\frac{27+20}{36} = \frac{47}{36} = 1\frac{11}{36}$ sum reqd.

Ex. 6. $\frac{2}{7} = \frac{4}{14}$, and $\frac{4+5}{14} = \frac{9}{14}$ sum of the given fractions.

Ex. 7. $2 \times 5 \times 7 = 70$ First numerator.
$3 \times 3 \times 7 = 63$ Second numerator.
$5 \times 3 \times 5 = 75$ Third numerator.
And $3 \times 5 \times 7 = 105$ Common denominator.
But $\frac{70 + 63 + 75}{105} = 1\frac{103}{105}$. Ans.

(Key to Vol. I. page 62.)

Ex. 8. $5 \times 5 \times 6 = 150$ First numerator.
$3 \times 9 \times 6 = 162$ Second numerator.
$1 \times 9 \times 5 = 45$ Third numerator.
And $9 \times 5 \times 6 = 270$ Common denominator.
But $2 + \frac{150+162+45}{270} = 3\frac{20}{90}$ the sum required.

Ex. 9. $\frac{3}{5} + \frac{4}{5}$ of $\frac{1}{3} = \frac{13}{15}$. Reducing this fraction and $\frac{3}{20}$ to a common denominator, the new fractions are $\frac{52}{60}$ and $\frac{9}{60}$.
But $9 + \frac{52+9}{60} = 10\frac{1}{60}$ sum required.

Ex. 10. $£\frac{2}{3} \times \overset{s.}{\frac{20}{1}} = \overset{s.}{\frac{40}{3}} = \frac{120}{9}sh.$

And $\overset{s.}{\frac{120+5}{9}} = \overset{s.}{\frac{125}{9}}sh. = \overset{s.}{13} .. \overset{d.}{10} .. \overset{qr.}{2\frac{2}{3}}$ Ans.

Ex. 11. $\overset{s.}{\frac{3}{5}} \times \overset{d.}{\frac{12}{1}} = \overset{d.}{\frac{36}{5}} = \overset{d.}{\frac{108}{15}}$. And $\overset{d.}{\frac{108+4}{15}} = \overset{d.}{\frac{112}{15}} = \overset{d.}{7} .. \overset{qr.}{1\frac{13}{15}}$. Ans.

Ex. 12. $£\frac{1}{7} \times \overset{s.}{\frac{20}{1}} = \overset{s.}{\frac{20}{7}}$. And $\frac{5}{12}d. \times \frac{1}{12}s.\ \frac{5}{144}sh.$ Then

$$\frac{\overset{s.}{20} \times 9 \times 144 + \overset{s.}{2} \times 7 \times 144 + \overset{s.}{5} \times 7 \times 9}{7 \times 9 \times 144} = \overset{s.}{\frac{3139}{1008}} = \overset{s.}{3} .. \overset{d.}{1} .. \overset{qr.}{1\frac{10}{21}}.\ \text{Ans.}$$

SUBTRACTION OF VULGAR FRACTIONS.

(Page 63.)

Quest. 3. $\frac{7-5}{12} = \frac{2}{12} = \frac{1}{6}$. Ans.

Quest. 4. $\frac{3}{13} = \frac{9}{39}$, and $\frac{9-4}{39} = \frac{5}{39}$. Ans.

(Key to Vol. I. page 63.)

Quest. 5. $\frac{5}{12} = \frac{65}{156}$, and $\frac{7}{13} = \frac{84}{156}$. But $\frac{84-65}{156} = \frac{19}{156}$. Ans.

Quest. 6. $\frac{2}{7}$ of $4\frac{1}{6} = 1\frac{4}{21} = 1\frac{32}{168}$, and $\frac{3}{8} = \frac{63}{168}$.

But $5\frac{63}{168} - 1\frac{32}{168} = 4\frac{31}{168}$ Ans.

Quest. 7. $\overset{s.}{\frac{2}{3}} \times \overset{s.}{\frac{3}{4}} = \frac{1}{2}sh. = \frac{9}{18}sh.$ and $£\frac{5}{9} \times \overset{s.}{\frac{20}{1}} = \overset{s.}{\frac{100}{9}} = \overset{s.}{\frac{200}{18}}$.

But $\frac{\overset{s.}{200} - \overset{s.}{9}}{18} = \frac{191}{18}sh. = \overset{s.}{10} \,..\, \overset{d.}{7} \,..\, \overset{qr.}{1\frac{1}{3}}$. Ans.

Quest. 8. $\frac{2}{7}$ of $£5\frac{1}{6} = £1\frac{20}{21}$, and $\overset{s.}{\frac{3}{5}} \times £\frac{1}{20} = £\frac{3}{100}$.

But $£1\frac{20}{21} - £\frac{3}{100} = £1\frac{937}{2100} = £1 \,..\, \overset{s.}{8} \,..\, \overset{d.}{11\frac{3}{53}}$. Ans.

MULTIPLICATION OF VULGAR FRACTIONS.

Ex. 3. $\frac{2}{7} \times \frac{5}{8} = \frac{1}{7} \times \frac{5}{4} = \frac{5}{28}$ Product required.

Ex. 4. $\frac{4}{15} \times \frac{5}{24} = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$ Product required.

Ex. 5. $\frac{3}{7} \times \frac{4}{9} \times \frac{14}{15} = \frac{1}{1} \times \frac{4}{9} \times \frac{2}{5} = \frac{8}{45}$. Ans.

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Ex. 6. $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{1} = \frac{1}{1} \times \frac{1}{1} \times \frac{1}{1} = \frac{1}{1} = 1$. Ans.

Ex. 7. $\frac{7}{9} \times \frac{3}{5} \times \frac{61}{14} = \frac{1}{3} \times \frac{1}{5} \times \frac{61}{2} = \frac{61}{30} = 2\frac{1}{30}$. Prod. reqd.

(Key to Vol. I. page 64.)

Ex. 8. $\frac{5}{6}\times\frac{2}{3}\times\frac{6}{7}=\frac{5}{1}\times\frac{2}{3}\times\frac{1}{7}=\frac{10}{21}$. Ans.

Ex. 9. $\frac{6}{1}\times\frac{2}{3}\times\frac{5}{1}=\frac{2}{1}\times\frac{2}{1}\times\frac{5}{1}=\frac{20}{1}=20$ Ans.

Ex. 10. $\frac{2}{9}\times\frac{3}{5}\times\frac{5}{8}\times\frac{23}{7}=\frac{1}{3}\times\frac{1}{1}\times\frac{1}{4}\times\frac{23}{7}=\frac{23}{84}$. Product.

Ex. 11. $\frac{23}{7}\times\frac{146}{33}=\frac{3358}{231}=14\frac{124}{231}$. Product required.

Ex. 12. $\frac{5}{1}\times\frac{2}{3}\times\frac{2}{7}\times\frac{3}{5}\times\frac{25}{6}=\frac{1}{1}\times\frac{2}{3}\times\frac{1}{7}\times\frac{1}{1}\times\frac{25}{1}=\frac{50}{21}=2\frac{8}{21}$ Ans.

DIVISION OF VULGAR FRACTIONS.

Ex. 3. $\frac{4)\,16\,(4}{5)\,25\,(5}$ *Quotient.*

Ex. 4. $\frac{3}{4}\Big)\frac{7}{16}\Big(\frac{28}{48}=\frac{7}{12}$ *Quotient.*

Ex. 5. $\frac{7}{6}\Big)\frac{14}{9}\Big(\frac{84}{63}=1\frac{1}{3}$ *Quotient.*

Ex. 6. $\frac{15}{7}\Big)\frac{5}{6}\Big(\frac{35}{90}=\frac{7}{18}$ *Quotient.*

Ex. 7. $\frac{3)\,12\,(4}{5)\,35\,(7}$ *Quotient.*

Ex. 8. $\frac{3}{5}\Big)\frac{2}{7}\Big(\frac{10}{21}$ *Quotient.*

(Page 65.)

Ex. 9. $\frac{3}{1}\Big)\frac{9}{16}\Big(\frac{3}{16}$ *Quotient.*

(Key to Vol. I. page 65.)

Ex. 10. $\frac{2}{1}\Big)\frac{3}{5}\Big(\frac{3}{10}$ *Quotient.*

Ex. 11. $\frac{86}{9}\Big)\frac{22}{3}\Big(\frac{198}{258} = \frac{33}{43}$ *Quotient.*

Ex. 12. $\frac{38}{7}\Big)\frac{2}{9}\Big(\frac{14}{342} = \frac{7}{171}$ *Quotient.*

RULE OF THREE IN VULGAR FRACTIONS.

Quest. 2. As $\overset{\text{oz.}}{\frac{1}{1}} : \overset{\text{oz.}}{\frac{27}{8}} :: \pounds 0 .. \overset{s.}{6} .. \overset{d.}{8} = \pounds\frac{1}{3} : \pounds\frac{27}{24} = \pounds 1 .. \overset{s.}{1} .. \overset{d.}{4\frac{1}{2}}$. Ans.

Quest. 3. As $\frac{6}{32} : \frac{5}{32}$, that is, As 6 : 5, $\pounds 273 .. \overset{s.}{2} .. \overset{d.}{6} : \pounds 227 .. \overset{s.}{12} .. \overset{d.}{1}$. Ans.

Quest. 4. As $\pounds\frac{100}{1} : \pounds 108\frac{5}{8} :: \pounds 1230 : \pounds 1336 .. \overset{s.}{1} .. \overset{d.}{9}$. Ans.

Quest. 5. As $\pounds\frac{100}{1} : \pounds 3\frac{1}{4} :: \pounds 273\frac{3}{4} : \pounds 8 .. \overset{s.}{17} .. \overset{d.}{11\frac{1}{4}}$. Ans.

Quest. 6. As $\pounds 73\frac{1}{16} : \pounds 250\frac{1}{2} :: \frac{1}{8} : \frac{3}{7}$. Ans.

Quest. 7. As $\overset{\text{inches.}}{7\frac{3}{4}} : \overset{\text{in.}}{12} :: \overset{\text{in.}}{12} : \overset{\text{in.}}{18\frac{18}{31}}$. Ans.*

Quest. 8. As $\overset{\text{yd.}}{\frac{3}{4}} : \overset{\text{yds.}}{2\frac{1}{2}}$, that is, As 3 : 10, $:: \overset{\text{yds.}}{9\frac{1}{2}} : \overset{\text{yds.}}{31\frac{2}{3}}$. Ans.

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* In some Editions of the *Course* (and perhaps in the last) there is an Error in the Answer.

(Page 66.)

Quest. 9. As $\overset{\text{sh.}}{8\frac{1}{2}} : \overset{\text{sh.}}{5} :: \overset{\text{oz.}}{6\frac{9}{10}} : \overset{\text{oz.}}{4\frac{1}{17}}$. Ans.

Quest. 10. As $\overset{\text{po.}}{11\frac{11}{12}} : \overset{\text{po.}}{40} :: \overset{\text{po.}}{4} : \overset{\text{po.}}{13\frac{61}{143}}$. Ans.

Quest. 11. As $\overset{\text{hrs.}}{11\frac{9}{10}} : \overset{\text{hrs.}}{13\frac{5}{8}} :: \overset{\text{days.}}{35\frac{1}{2}} : \overset{\text{days.}}{40\frac{615}{952}}$. Ans.

Quest. 12. $\overset{\text{yd.}}{\frac{7}{8}} : \overset{\text{yd.}}{\frac{13}{8}}$, that is, As 7 : 13, $:: \overset{\text{yds.}}{2\frac{1}{2}} : \overset{\text{yds.}}{\frac{65}{14}} = 4\frac{9}{14}$ yards.

But $\overset{\text{yds.}}{4\frac{9}{14}} \times 976 = \overset{\text{yds.}}{4531\frac{3}{7}} = \overset{\text{yds.}}{4531} .. \overset{\text{qr.}}{1} .. \overset{\text{nls.}}{2\frac{6}{7}}$. Ans.

(Key to Vol. I. page 67.)

DECIMAL FRACTIONS.

ADDITION OF DECIMALS.

Ex. 2.
276,
39,213
72014,9
417,
5032,
Sum 77779,113

Ex. 3.
7530,
16,201
3,0142
957,13
6,72119
,03014
Sum 8513,09653

Ex. 4. Ans. 17500,9718.

SUBTRACTION OF DECIMALS.

(Page 68.)

Ex. 2.
From 2,73
Take 1,9185
0,8115 Rem.

Ex. 3.
From 214,81
Take 4,90142
Rem. 209,90858

Ex. 4.
From 2714,
Take ,916
Rem. 2713,084

MULTIPLICATION OF DECIMALS.

(Page 69.)

Ex. 2.
Multiply 79,347
by 23,15
Product 1836,88305

Ex. 3.
Multiply ,63478
by ,8204
Product ,520773512

(Key to Vol. I. page 69.)

Ex. 4. Multiply ,385746
by ,00464

,00178986144 Product.

CONTRACTION I.

Ex. 2. 2·714 × 100 = 271·4 Product required.

Ex. 3. ·916 × 1000 = 916 Product required.

Ex. 4. 21·31 × 10000 = 213100 Product required.

CONTRACTION II.

(Page 70.)

Ex. 2. 480,14936 Multiplicand.
61427·2 Multiplier reversed.

9602987
3361045
96030
19206
480
288

1308·0036 Product required.

Ex. 3. 2490,30480 Multiplicand.
682375·0 Multiplier reversed.

124515240
17432134
747091
49806
19922
1494

1427·65687 Product required.

(Key to Vol. I. page 70.)

Ex. 4. Ans. 235·104|0907965204.

DIVISION OF DECIMALS.

(Page 71.)

Ex. 3. 54·25)123·70536(2·2802 Quotient.

Ex. 4. ·7854)12·0000000(15·278 Quotient.

Ex. 5. 100)4195·68(41·9568 Quotient.

Ex. 6. ·153)·8297592(5·4232 Quotient.

CONTRACTION I.

(Page 72.)

Ex. 2. Quotient 1·281875

Ex. 3. Quotient ·04412037037 &c.

Ex. 4. Quotient ·0007721519 *nearly.*

CONTRACTION II.

Ex. 2. 100)5·16(·0516 Quotient.

Ex. 3. 10)419(41·9 Quotient.

Ex. 4. 1000)·21(·00021 Quotient.

CONTRACTION III.

(Page 73.)

Ex. 2. 230·409)41092351(17·8345 Quotient.
180514
19228
796
105
13
2

(Key to Vol. I. page 73.)

Ex. 3. 5713·9)37·10438(·00649 Quotient.

Ex. 4. 213,)91·308(·427 Quotient.

REDUCTION OF DECIMALS.

Ex. 2.
$$\begin{array}{ll} & 4)1{\cdot}00({\cdot}25 \\ & 2)1{\cdot}0({\cdot}5 \\ \text{And} & 4)3{\cdot}00({\cdot}75 \end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\}\text{Therefore }{\cdot}25,\ {\cdot}5,\text{ and }{\cdot}75.\quad\text{Ans.}$$

Ex. 3. 8)5·000(·625 the decimal required.

Ex. 4. 25)3·00(·12 decimal required.

Ex. 5. 192)6·000(·03125 Ans.

Ex. 6. 3842)550·0(·143154 &c. decimal required.*

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* This Answer differs in the last figure from the Answer given by Dr. Hutton.

(Page 74.)

To find the Value of a Decimal in terms of the Inferior Denominations.

Ex. 2. $\overset{s.}{{\cdot}625} \times \overset{d.}{12} = \overset{d.}{7\tfrac{1}{2}}$. Ans.

Ex. 3. $£{\cdot}8635 \times \overset{s.}{20} = \overset{s.}{17{\cdot}27}$, and $\overset{s.}{{\cdot}27} \times \overset{d.}{12} = \overset{d.}{3{\cdot}24}$.

Therefore $£0 \,..\, \overset{s.}{17} \,..\, \overset{d.}{3{\cdot}24}$ Ans.

Ex. 4. $\overset{\text{lb.}}{{\cdot}0125} \times \overset{\text{dwt.}}{240} = 3\,\text{dwt.}$ Ans.

Ex. 5.
$$\begin{array}{rr} & \overset{\text{lb.}}{{\cdot}4694} \times \overset{\text{oz.}}{12} = \overset{\text{oz.}}{5{\cdot}6328} \\ & 20 \\ \hline \text{dwt.} & 12{\cdot}6560' \\ & 24 \\ \hline \text{gr.} & 15{\cdot}744 \\ \hline \end{array}$$

Therefore $\overset{\text{oz.}}{5} \,..\, \overset{\text{dwt.}}{12} \,..\, \overset{\text{gr.}}{15.744}$. Ans.

(Key to Vol. I. page 74.)

Ex. 6. $\overset{\text{cwt.}}{\cdot625} \times \overset{\text{qr.}}{4} = \overset{\text{qr.}}{2\cdot5} = \overset{\text{qr.}}{2} \,..\, \overset{\text{lb.}}{14}.$ Ans.

Ex. 7.
$$\begin{array}{rr} \overset{\text{miles.}}{\cdot009943} \times \overset{\text{yds.}}{1760} = & \overset{\text{yds.}}{17\cdot49968} \\ & 3 \\ \cline{2-2} \text{Feet} & 1\cdot49904 \\ & 12 \\ \cline{2-2} \text{Inches} & 5\cdot98848 \\ & 3 \\ \cline{2-2} \text{Barleycorns} & 2\cdot96544 \\ \cline{2-2} \end{array}$$

Wherefore $\underline{\overset{\text{yds.}}{17} \,..\, \overset{\text{ft.}}{1} \,..\, \overset{\text{in.}}{5} \,..\, \overset{\text{b. corns}}{2\cdot96544}}$ Ans.

Ex. 8. $\overset{\text{yds.}}{\cdot6875} \times \overset{\text{qrs.}}{4} = \overset{\text{qrs.}}{2\cdot75} = \overset{\text{qrs.}}{2} \,..\, \overset{\text{nls.}}{3}.$ Ans.

Ex. 2.
$$\begin{array}{rr} \overset{\text{acres.}}{\cdot3375} \times \overset{\text{roods.}}{4} = & \overset{\text{ro.}}{1\cdot35} \\ & 40 \\ \cline{2-2} \text{Perches} & 14\cdot00 \\ \cline{2-2} \end{array}$$

Therefore $\underline{\overset{\text{ro.}}{1} \,..\, \overset{\text{po.}}{14}.}$ Ans.

Ex. 10. $\overset{\text{hhds.}}{\cdot2083} \times \overset{\text{gal.}}{63} = \overset{\text{gallons}}{13\cdot1229}$ Ans.

(Page 75.)

To reduce Integers or Decimals to Equivalent Decimals of higher Denominations.

Ex. 2. $\overset{\text{d.}}{9} \div (12 \times 20) = £\cdot0375$ the decimal required.

Ex. 3. $\overset{\text{dr.}}{7} \div (16 \times 16) = \overset{\text{lb.}}{\cdot02734375}$ Ans.

Ex. 4. $\overset{\text{d.}}{\cdot26} \div (12 \times 20) = £\cdot0010833$ &c. the decimal reqd.

(Key to Vol. I. page 75.)

Ex. 5. $\overset{\text{lb.}}{2{\cdot}15} \div (28 \times 4) = \overset{\text{cwt.}}{{\cdot}019196}$ &c. the decimal required.

Ex. 6. $\overset{\text{yds.}}{24} \div 1760 = \overset{\text{miles.}}{{\cdot}013636}$ &c. Ans.

Ex. 7. $\overset{\text{poles.}}{{\cdot}056} \div (40 \times 4) = \overset{\text{acres.}}{{\cdot}00035}$ the decimal required.

Ex. 8. $\overset{\text{pt.}}{1{\cdot}2} \div (8 \times 63) = \overset{\text{hhd.}}{{\cdot}00238}$ &c. Ans.

Ex. 9. $\overset{\text{min.}}{14} \div (60 \times 24) = \overset{\text{day}}{{\cdot}009722}$ &c. decimal required.

Ex. 10. $\overset{\text{pt.}}{{\cdot}21} \div 16 = \overset{\text{peck}}{{\cdot}013125}$ Ans.

Ex. 11.* $28'' .. 12''' = \frac{28}{60} + \frac{12}{60} \text{ of } \frac{1}{60}$ minute.

Also $\overset{\text{min.}}{28} \div 60 = {\cdot}466\dot{6}$ minute.

And $\overset{\text{min.}}{12} \div 3600 = 003\dot{3}$ minute.

Sum of both decimals ·47′ the decimal required.

Otherwise,

$28'' .. 12''' = 1692''', = \frac{1692}{3600}$ minute $= {\cdot}47'$ *as before.*

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* See this question solved the 6th in the next article.

When there are several Numbers to be reduced Collectively to the Decimal of the highest Denomination.

(Page 76.)

Ex. 2.

$$\begin{array}{r|l} \text{qr. } 4 & 1{\cdot} \\ d.\ 12 & 3{\cdot}25 \\ s.\ 20 & 17{\cdot}27083\dot{3} \\ \multicolumn{2}{r}{\pounds\ 19{\cdot}863541\dot{6}} \end{array}$$

decimal required.

Ex. 3.

$$\begin{array}{r|l} d.\ 12 & 6{\cdot} \\ s.\ 20 & 15{\cdot}5 \\ \multicolumn{2}{r}{\pounds\ \ 0{\cdot}775} \end{array}$$

decimal required.

(Key to Vol. I. page 76.)

Ex. 4. qr. 4 | 2·
d. 12 | 7·5
sh. 0·625 decimal required.

Ex. 5. gr. 24 | 16·
dwt. 20 | $12{\cdot}66\dot{6}$
oz. 12 | $5{\cdot}633\dot{3}$
lb. 0·46944. Ans.

Ex. 6. *(Referred to, above.)*
thirds 60 | 12‴.
seconds 60 | 28·2″.
Minute 0·47′. Ans. [*See Ex.* 11. *last article.*]

DUODECIMALS.

(Page 77.)

Ex. 3.
ft. in.
4 .. 7
9 .. 6
41 .. 3
2 .. 3 .. 6
Sq. feet. 43 .. 6 .. 6 Prod.

Ex. 4.
ft. in.
12 .. 5
6 .. 8
74 .. 6
8 .. 3 .. 4
Sq. ft. 82 .. 9 .. 4 Prod.

Ex. 5.
ft. in. 12ths.
35 .. 4 .. 6
12 .. 3
424 .. 6 .. 0
8 .. 10 .. 1 .. 6
Sq. feet 433 .. 4 .. 1 .. 6 Prod.

Ex. 6.
ft. in. 12ths.
64 .. 6 .. 0
8 .. 9 .. 3
516 .. 0 .. 0
48 .. 4 .. 6 .. 0
1 .. 4 .. 1 .. 6 .. 0
Sq. ft. 565 .. 8 .. 7 .. 6 .. 0 Pro.

⁎ In this Rule the second term of the Product, to adapt it to the capacity of workmen, is generally denominated inches; but it is requisite the student should clearly understand that inches con-

(Key to Vol. I. page 77.)

stitute the *third* term in Square Measure, and the *fourth* in Solid; of which last species, however, the author of the COURSE has given no example.

Now it is only necessary to remember that, in Decimals, Duodecimals, Time, and Degrees, the value decreases *gradatim* in Geometrical Progression from the left hand to the right; and that the common ratio in Decimals is 10, in Duodecimals 12, and in Time and Degrees 60. Hence the three series decreasing from unity are,

	0.	1.	2.	3.	4.	5. *Terms.*
Decimals.	$\frac{1}{1}$,	$\frac{1}{10}$,	$\frac{1}{100}$,	$\frac{1}{1000}$,	$\frac{1}{10000}$,	&c. .....
Duodecimals.	$\frac{1}{1}$,	$\frac{1}{12}$,	$\frac{1}{144}$,	$\frac{1}{1728}$,	&c.	
Time and Degrees.	$\frac{1}{1}$,	$\frac{1}{60}$,	$\frac{1}{3600}$,	$\frac{1}{216000}$,	&c. called also	

minutes, seconds, thirds, fourths, fifths, &c.

The object of Duodecimals is to ascertain, by an easy method, the number of square, or of solid feet, in a piece of work or other body to which the Rule is applicable; and it is evident, that in multiplying length by breadth, feet into feet, give *square feet;* and inches into inches, *square inches:* feet, therefore, into inches must give 12*th parts of* SQUARE *feet for the second term* of the Product. In the subsequent pages of the KEY, when admeasurement of this kind occurs, we shall place *pts.* (meaning 12*th* parts of sqr. feet) before inches; as 193 sq. ft. 10 pts. 7 inches, and in the meantime it is thought a further elucidation to subjoin an example of Solid Measure, as very proper to come under the present Rule.

Required the solidity of a wall 130 *ft.* .. 8 *in.* *long,* 10 *ft.* .. 4 *in.* *high, and* 2 *ft.* .. 3 *in.* *in thickness throughout.*

ft. in.
130 .. 8 long.
Multiplier 10 .. 4 high.

Sq^{r}. feet 1306 .. 8 Prod. of the length by 10 feet high.
43 .. 6 .. 8 Prod. of the length by 4 inches high.

Sq^{r}. feet 1350 .. 2 .. 8 area of either side of the wall.
Multiplier 2 .. 3 .. 0 thickness of the wall.

Solid feet 2700 .. 5 .. 4 Prod. of the area by 2 feet thick.
337 .. 6 .. 8 .. 0 Prod. of the area by 3 inches thick.

Cubic feet 3038 .. 0 .. 0 .. 0 the content required.

(Key to Vol. I. page 79.)

INVOLUTION.

Quest. 1. $45 \times 45 = 2025$. Ans.

Quest. 2. $4{\cdot}16 \times 4{\cdot}16 = 17{\cdot}3056$. Ans.

Quest. 3. $3{\cdot}5 \times 3{\cdot}5 \times 3{\cdot}5 = 42{\cdot}875$. Ans.

Quest. 4. ${\cdot}029 \times {\cdot}029 \times {\cdot}029 \times {\cdot}029 \times {\cdot}029 = {\cdot}000000020511149$. Ans.

Quest. 5. $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ Ans.

Quest. 6. $\frac{5}{9} \times \frac{5}{9} \times \frac{5}{9} = \frac{125}{729}$. Ans.

Quest. 7. $\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{81}{256}$. Ans.

EVOLUTION.

(Page 83.)

Quest. 3.

$$\begin{array}{r l} & 2\dot{0}2\dot{5}(45. \quad \text{Ans.} \\ & 16 \\ 85) & \overline{425} \\ & \underline{425} \\ & \cdots \end{array}$$

Quest. 4.

$$\begin{array}{r l} & 1\dot{7}{\cdot}3\dot{0}5\dot{6}(4{\cdot}16 \quad \text{Root required.} \\ & 16 \\ 81) & \overline{130} \\ & \underline{81} \\ 826) & 4956 \\ & \underline{4956} \\ & \cdots\cdot \end{array}$$

Quest. 5.

$$\begin{array}{r l} & {\cdot}0\dot{0}0\dot{7}2\dot{9}({\cdot}027 \quad \text{Root required.} \\ & \underline{4} \\ 47) & 329 \\ & \underline{329} \\ & \cdots \end{array}$$

(Key to Vol. I. page 83.)

Quest. 6. $\dot{3}\cdot0\dot{0}0\dot{0}0\dot{0}$(1·73205 &c. Ans.

Quest. 7. $\dot{5}\cdot0\dot{0}0\dot{0}$ &c. (2·236068 &c. Ans.

Quest. 8. $\dot{6}\cdot0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}$(2·449489. Ans.

3638879 Remainder.

Quest. 9. $\dot{7}\cdot0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}$(2·645751 Root required.

1645999 Remainder.

Quest. 10. $1\dot{0}\cdot0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}$(3·162277. Ans.

4175271 Remainder.

Quest. 11. $1\dot{1}\cdot0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}$(3·316624 Root required.

5242624 Remainder.

Quest. 12. $1\dot{2}\cdot0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}$(3·464101 Root required.

4261799 Remainder.

(Page 84.)

SQUARE ROOT OF VULGAR FRACTIONS, AND MIXED NUMBERS.

Quest. 1. $\sqrt{\frac{25}{36}} = \frac{\sqrt{25}}{\sqrt{36}} = \frac{5}{6}$. Ans.

Quest. 2. $\sqrt{\frac{27}{147}} = \frac{\sqrt{27}}{\sqrt{147}} = \frac{\sqrt{9}}{\sqrt{49}} = \frac{3}{7}$ Root required.

(Key to Vol. I. page 84.)

Quest. 3. $\sqrt{\frac{9}{12}} = \sqrt{\cdot 75} = \cdot 8660254$ &c. Ans.

Quest. 4. $\sqrt{\frac{5}{12}} = \sqrt{\cdot 4166\dot{6}} = \cdot 645497$ &c. Root reqd.

Quest. 5. $\sqrt{17\frac{3}{8}} = \sqrt{17\cdot 3750} = 4\cdot 16833\dot{3}$. Ans.

The 4th, 8th, 16th, 32nd, &c. Root, being an Integer Power of 2.

Quest. 2. $\sqrt[4]{97\cdot 41} = \sqrt[2]{\sqrt[2]{97.41}} = \sqrt{9\cdot 86965}$ &c. $= 3\cdot 1416$ *very nearly.* Ans.

CUBE ROOT.

(Page 86.)

Ex. 2.

$$\begin{array}{r|l} 3\times 8^2 = 192 & 57\dot{1}48\dot{2}\cdot\dot{1}90(82\cdot 9\text{ \&c. Root reqd.} \\ 3\times 8 = 24 & 512 \\ \hline \text{1st Divisor } 1944) & 59482 \end{array}$$

$$\left\{\begin{array}{l} 3\times 8^2\times 2 = ----384 \\ 3\times 8\times 2^2 = -----96 \\ \phantom{3\times 8\times{}} 2^3 = --------8 \end{array}\right\}\text{add}$$

$$\begin{array}{r|r} 3\times 82^2 = 20172 & \\ 3\times 82 = 246 & 39368\text{ Subtrahend.} \\ \hline \text{2d Div}^{r}.\ 201966\ ----------) & 20114190\text{ Resolvend.} \end{array}$$

$$\left\{\begin{array}{l} 3\times 82^2\times 9 = ---181548 \\ 3\times 82\times 9^2 = -----19826 \\ \phantom{3\times 82\times{}} 9^3 = --------729 \end{array}\right\}\text{add}$$

$$\begin{array}{r} 18353789\text{ Subtrahend.} \\ \hline 1760401\text{ Remainder.} \\ \hline \end{array}$$

Quest. 3. Ans. 11·76 and 1778424 Remainder.

Quest. 4. Ans. 11·0027 and 19659410317 Remaining.

(Key to Vol. I. page 88.)

EXTRACTION OF THE CUBE ROOT BY APPROXIMATION.

Ex. 2. Trial soon determines the Cube Root of ·67 to be between ·87 and ·88. Taking, therefore, ·88, its Cube is ·681472 *the assumed Cube.*

Then ·681472
2

Double the assumed Cube - - 1·362944
Given number - - - - ·67

Sum of the given number and double the assumed Cube - - - } 2·032944 = A.

·67
2

Double the given number 1·34
Assumed Cube ·681472

Sum of the assumed Cube and double the given number- - - } 2·021472 = B.

(A) (B) assu^d root.
And, As 2·032944 : 2·021472 : : ·88 : ·8750346 Root, *nearly.*

Ex. 3. Trial determines the Root to be near ·22.
But the Cube of ·22 is ·010648 *the assumed Cube.*
(A) (B) assu^d root.
And, As ·031296 : ·030648 : : ·22 : ·215444 Root, *nearly.*

GENERAL ROOTS.

(Page 89.

THE EXTRACTION OF ANY ROOT WHATEVER.

Quest. 1. Here $P = 2$
$n = 3$
$r = 1\cdot2$
$A = 1\cdot728$

(Key to Vol. I. page 89.)

$$\begin{array}{llll@{\qquad}lll} \text{And} & 6{\cdot}912 & = n+1 \text{ times } A. & & \text{Also} & 8 & = n+1 \text{ times } P.\\ & 4{\cdot} & = n-1 \text{ times } P. & & & 3{\cdot}456 & = n-1 \text{ times } A.\\ \cline{2-2} \cline{6-6} & 10{\cdot}912 \text{ sum.} & & & & 11{\cdot}456 \text{ sum.} & \end{array}$$

But, As $10{\cdot}912 : 11{\cdot}456 :: 1{\cdot}2\ (=r) : 1{\cdot}25982$. First approximation to the root required.

Proceeding now with 1·25982 as r, the result is,

1·259921 the Root, *nearly.*

Quest. 2. Here $P = 3214$
$n = 3$
$r = 14{\cdot}7$
$A = 3176{\cdot}523$, whence 14·75758. Ans.

Quest. 3. $\sqrt[4]{2} = \sqrt[2]{\sqrt[2]{2}} = \sqrt{1{\cdot}4142136} = 1{\cdot}189207$ &c. Ans.

Quest. 4. Here $P = 97{\cdot}41$
$n = 4$
$r = 3{\cdot}1416$
$R = 3{\cdot}1415999$, which was required.*

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* See the last Quest. page 84, vol. i. of the *Course*, and its Answer in the KEY.

Quest. 5. $P = 2$
$n = 5$
$r = 1{\cdot}14$
$A = 1{\cdot}9254145824$.

And, As $19{\cdot}5524874944 : 19{\cdot}7016583296 :: 1{\cdot}14 (=r) : 1{\cdot}148697$ first approximation to the 5th Root of 2. Proceeding next with 1·148697 as r, the result is 1·148699 &c. Root, *nearly.*

Quest. 6. $\sqrt[6]{21035{\cdot}8} = \sqrt[3]{\sqrt[2]{21035{\cdot}8}}$ or
$\sqrt[2]{\sqrt[3]{21035{\cdot}8}} = \sqrt{27{\cdot}60491056} = 5{\cdot}254037$. Ans.

Quest. 7. $\sqrt[6]{2} = \sqrt[2]{\sqrt[3]{2}} = \sqrt{1{\cdot}259921} = 1{\cdot}122462$. Ans.

Or, by the Rule for GENERAL Roots,
As $(n+1)\frac{1}{2}A + (n-1)\frac{1}{2}P : P \backsim A :: r : R \backsim r$. That is,
As $11{\cdot}5464509325635 : {\cdot}129585447839 :: 1{\cdot}11 : {\cdot}012457 = R \backsim r$.

$$\begin{array}{rl} \text{Add, Root of the assumed Power,} & 1{\cdot}11 \\ \cline{2-2} \text{Root } (\textit{nearly}) & 1{\cdot}122457 \\ \cline{2-2} \end{array}$$

And 1·122457 (*the Root last found*) is, by another approximation, discovered to be ·000005 too little, consequently 1·122462 Root, *as before.* Ans.

(Key to Vol. I. page 89.)

Quest. 8. $\text{P} = 21035{\cdot}8$
$n = 7{\cdot}$
$r = 4{\cdot}$
$\text{A} = 16384{\cdot}$, whence $4{\cdot}1$ is the 1st approximation.

But $\overline{4{\cdot}1}\rvert^{7} = 19475{\cdot}4273881$, whereby is obtained $4{\cdot}145392$ the Root, *nearly*.

Quest. 9. $\text{P} = 2$
$n = 7$
$r = 1{\cdot}1$
$\text{A} = 1{\cdot}7777771$

From which, by two approximations, is found $1{\cdot}104089$. Ans.

Quest. 10. $\sqrt[8]{21035{\cdot}8}=\sqrt[2]{\sqrt[2]{\sqrt[2]{21035{\cdot}8}}}=3{\cdot}470323$. Root reqd.

Quest. 11. $\sqrt[8]{2}=\sqrt[2]{\sqrt[2]{\sqrt[2]{2}}}=\sqrt[2]{\sqrt[4]{2}}=\sqrt{1{\cdot}189207}=1{\cdot}090508$. Ans.

Quest. 12. $\sqrt[9]{21035{\cdot}8}=\sqrt[3]{\sqrt[3]{21035{\cdot}8}}=\sqrt[3]{27{\cdot}60491056}=3{\cdot}022239$. Ans.

Quest. 13. $\sqrt[9]{2}=\sqrt[3]{\sqrt[3]{2}}=\sqrt[3]{1{\cdot}259921}=1{\cdot}080059$. Ans.

ARITHMETICAL PROPORTION.

(Page 112.)

PROB. I.

Quest. 2. $\dfrac{(12+1)\times 12}{2} = 78$. Ans.

(Page 113.)

Quest. 3. $\dfrac{(24+1)\times 24}{2} = 300$. Ans.

Quest. 4. $\dfrac{(103+1)\times 52}{2} = 2704s. = \pounds 135 \,..\, \overset{s.}{4}$. Ans.

PROB. II.

Quest. 2. $\dfrac{70-10}{20} = 3$ the common difference.

And $\dfrac{(70+10)\times 21}{2} = 840$ the sum of the series.

(Key to Vol. I. page 113.)

Quest. 3. $\frac{\overset{s.}{103} - \overset{s.}{1}}{52-1} = \frac{\overset{s.}{102}}{51} = 2.$ Ans.

(Page 114.)

PROB. III.

Quest. 2. $3 \times 20 = 60$ the difference of the extremes,
And $70 - 60 = 10$ the less extreme.
But $\frac{(70+10) \times 21}{2} = 840$ the sum of the series.
Therefore 10, and 840. Ans.

Quest. 3. $2 \times 51 = 102$ the difference of the extremes.
And $\overset{s.}{102} + \overset{s.}{1} = \overset{s.}{103}$ the greater extreme.
Also $\frac{(103+1) \times 52}{2} = \overset{s.}{2704} = £135 \,..\, 4$ the sum of the series.
Therefore £135 .. 4 the debt, and £5 .. 3 the last payment. Ans.

GEOMETRICAL PROGRESSION.

(Page 117.)

Quest. 1. $1 \times 512 = 512$ the greater extreme,
And $\frac{511 + 512}{2-1} = 1023$ the sum of the series.
Therefore 512, and 1023. Ans.

Quest. 2. $£1 \times £2048 = £2048$ the greater extreme,
and $\frac{£2048 + \overset{£}{2047}}{2-1} = £4095$ sum of the series.
Therefore £4095 the debt, and £2048 the last payment. Ans.

(Key to Vol. I. page 121.)

SINGLE FELLOWSHIP.

Quest. 3. As £120 : £30 : : £75 : £18 .. 15 s. = C's share,
and consequently £11 .. 5 = D's share.

Quest. 4.

		s.				s.	d.	qr.	
As £700 :	£125 ..	10 : :	£123 :	£22 ..		1 ..	0 ..	$2\frac{2}{35}$	= E's share.
700 :	125 ..	10 : :	358 :	64 ..		3 ..	8 ..	$0\frac{32}{35}$	= F's share.
700 :	125 ..	10 : :	219 :	39 ..		5 ..	3 ..	$1\frac{1}{35}$	= G's share.
				£125..10					

Quest 5.

				£.	£.	s.	d.	
250	are as	5	As 30 : 5 : :	700 :	116 ..	13 ..	4	....1st village.
350		7	As 30 : 7 : :	700 :	163 ..	6 ..	8	....2d village.
400		8	As 30 : 8 : :	700 :	186 ..	13 ..	4	....3d village.
500		10	As 30 : 10 : :	700 :	233 ..	6 ..	8	....4th village.
	Sum	30			£700			

Quest. 6.

£500	are as	£100
320		64
75		25
	Sum	£189

			A.	R.	P.	A.	R.	P.	
And	As £189 :	£100 : :	37 ..	2 ..	14 :	20 ..	3 ..	$39\frac{139}{179}$	....L's share.
	189 :	64 : :	37 ..	2 ..	14 :	13 ..	1 ..	$30\frac{46}{179}$	....M's share.
	189 :	25 : :	37 ..	2 ..	14 :	3 ..	0 ..	$23\frac{173}{179}$	....N's share.
						37 ..	2 ..	14	

Quest. 7.

£	s.	d.	
£ 57 ..	15 ..	0	
108 ..	3 ..	8	
22 ..	0 ..	10	
73 ..	0 ..	0	
£260 ..	19 ..	6	Sum. And,

(Key to Vol. I. page 121.)

	£.	s.	d.		£.	s.		£.	s.	d.		£.	s.	d.	qrs.	
As	260	19	6	:	170	14	: :	57	15	0	:	37	15	5	$2\frac{5302}{10439}$	*O.*
	260	19	6	:	170	14	: :	108	3	8	:	70	15	2	$2\frac{7498}{10439}$	*P.*
	260	19	6	:	170	14	: :	22	0	10	:	14	8	4	$0\frac{4720}{10439}$	*Q.*
	260	19	6	:	170	14	: :	73	0	0	:	47	14	11	$2\frac{3358}{10439}$	*R.*
												£ 170	14	0	0	

(Page 122.)

Quest. 8. £900 − £540 = £360 the loss sustained by S, T, and V.

$$\text{And}\left\{\begin{array}{lcrl} £360 \times \frac{1}{8} & = & £\ 45 & = \text{S's loss.} \\ 360 \times \frac{1}{4} & = & 90 & = \text{T's loss.} \\ 360 \times \frac{5}{8} & = & 225 & = \text{V's loss.} \\ & & \overline{£360} & \end{array}\right\}\text{Ans.}$$

Quest. 9.

$$\begin{array}{lcrr} & & s. & s.\quad d. \\ \frac{1}{2}\text{ of } 25 & = & & 12 \;..\; 6 \\ \frac{1}{3}\text{ of } 25 & = & & 8 \;..\; 4 \\ \frac{1}{4}\text{ of } 25 & = & & 6 \;..\; 3 \\ \frac{1}{5}\text{ of } 25 & = & & 5 \;..\; 0 \\ & & \text{Sum} & \overline{32 \;..\; 1} \end{array}$$

		s.	d.		s.		s.	d.		s.	d.	qr.		
And	As	32	1	:	25	: :	12	6	:	9	8	$3\frac{41}{77}$	W.	Ans.
		32	1	:	25	: :	8	4	:	6	5	$3\frac{53}{77}$	X.	
		32	1	:	25	: :	6	3	:	4	10	$1\frac{59}{77}$	Y.	
		32	1	:	25	: :	5	0	:	3	10	$3\frac{1}{77}$	Z.	
										25 *shillings.*				

Otherwise.

The fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, brought to a common denominator, are, $\frac{30}{60}$, $\frac{20}{60}$, $\frac{15}{60}$, and $\frac{12}{60}$; but the shares will be as the numerators of these new fractions, the sum of which is 77. That is,

	s.		s.		s.		s.	d.	qr.		
As	77	:	25	: :	30	:	9	8	$3\frac{41}{77}$	W.	Ans. *as before.*
	77	:	25	: :	20	:	6	5	$3\frac{53}{77}$	X.	
	77	:	25	: :	15	:	4	10	$1\frac{59}{77}$	Y	
	77	:	25	: :	12	:	3	10	$3\frac{1}{77}$	Z.	

(Key to Vol. I. page 122.)

	men.		men. men. men. men.	
Quest. 10.	54	As	228 : 76 : : 54 : 18	from the 1st Company.
	51		228 : 76 : : 51 : 17	------ 2nd Company.
	48		228 : 76 : : 48 : 16	------ 3rd Company.
	39		228 : 76 : : 39 : 13	------ 4th Company.
	36		228 : 76 : : 36 : 12	------ 5th Company.
	Sum 228		Proof 76 Men.	

DOUBLE FELLOWSHIP.

(Page 123.)

Quest. 2.
hors. da.
23 × 27 = 621 for C.
and 21 × 39 = 819 for D.

Now $\left.\begin{matrix}621\\819\end{matrix}\right\}$ are as $\left\{\begin{matrix}69\\91\end{matrix}\right.$

Sum 160

And { As 160 : 69 : : £54 : £23 .. 5 s. .. 9 d. ---- C. } Ans.
160 : 91 : : 54 : 30 .. 14 .. 3 ---- D. }

£54 .. 0 .. 0

Quest. 3.
ox. mo.
7 × 3 = 21 for E.
9 × 5 = 45 for F.
4 × 12 = 48 for G.

And $\left\{\begin{matrix}21\\45\\48\end{matrix}\right\}$ are as $\left\{\begin{matrix}7\\15\\16\end{matrix}\right.$

Sum 38

			s.	d.	qr.		
But	As 38 : £30 : : 7 :	£ 5 ..	10 ..	6 ..	$1\frac{5}{19}$	--- E.	Ans.
	38 : 30 : : 15 :	11 ..	16 ..	10 ..	$0\frac{8}{19}$	--- F.	
	38 : 30 : : 16 :	12 ..	12 ..	7 ..	$2\frac{6}{19}$	--- G.	
		£30 ..	0 ..	0 ..	0		

(Key to Vol. I. page 123.)

Quest. 4.
s. mo. off.
40 × 6 × 4 = 960 for the officers.
s. mo. pet. off.
30 × 6 × 12 = 2160 for the midshipmen.
s. mo. seam.
22 × 3 × 110 = 7260 for the sailors.

$$\text{And}\quad \left.\begin{array}{r}960\\2160\\7260\end{array}\right\}\text{ are as }\left\{\begin{array}{r}16\\36\\121\end{array}\right.$$

Sum 173

But
$$\begin{array}{l}
\text{As }173 : \overset{\pounds}{1000} :: 16 : \pounds 92 \,..\, \overset{s.}{9} \,..\, \overset{d.}{8} \,..\, \overset{qr.}{2}\tfrac{132}{1038} \text{ - - - Officers.}\\
\phantom{\text{As }}173 : 1000 :: 36 : 208 \,..\, 1 \,..\, 10 \,..\, 0\tfrac{816}{1038} \text{ - - - Midshipmen.}\\
\phantom{\text{As }}173 : 1000 :: 121 : 699 \,..\, 8 \,..\, 5 \,..\, 1\tfrac{90}{1038} \text{ - - - Seamen.}
\end{array}$$

£1000 .. 0 .. 0 .. 0

And,

$$\left.\begin{array}{l}
\overset{\pounds}{}\;\; s. \;\; d. \;\; qr. \qquad\qquad \pounds \;\; s. \;\; d. \;\; qr.\\
4)\ 92 \,..\, 9 \,..\, 8 \,..\, 2\tfrac{132}{1038} = 23 \,..\, 2 \,..\, 5 \,..\, 0\tfrac{92}{173} \text{ for each Off}^{\text{r}}.\\
12)208 \,..\, 1 \,..\, 10 \,..\, 0\tfrac{816}{1038} = 17 \,..\, 6 \,..\, 9 \,..\, 3\tfrac{69}{173} \text{ - - - - Midsh.}\\
110)699 \,..\, 8 \,..\, 5 \,..\, 1\tfrac{90}{1038} = 6 \,..\, 7 \,..\, 2 \,..\, 0\tfrac{8}{173} \text{ - - - - Seam.}
\end{array}\right\}\text{Ans.}$$

(Page 124.)

Quest. 5.
mo.
£1000 × 12 = £12000 - - - - H.
1500 × 10 = 15000 - - - - I.
2800 × 7 = 19600 - - - - K.

$$\text{Now}\quad \left.\begin{array}{r}\pounds 12000\\15000\\19600\end{array}\right\}\text{ are as }\left\{\begin{array}{r}\pounds 60\\75\\98\end{array}\right.$$

Sum £233

(Key to Vol. I. page 124.)

And

$$\left.\begin{array}{lrrrrrrl}
 & & \text{s.} & & & \text{s.} & \text{d.} & \\
\text{As } \pounds 233 : & \pounds 1776\,.. & 10 :: & \pounds 60 : & \pounds 457\,.. & 9\,.. & 4\frac{1}{4} & \text{- - H.}\\
233 : & 1776\,.. & 10 :: & 75 : & 571\,.. & 16\,.. & 8\frac{1}{4} & \text{- - I.}\\
233 : & 1776\,.. & 10 :: & 98 : & 747\,.. & 3\,.. & 11\frac{1}{4} & \text{- - K.}
\end{array}\right\}\text{Ans.*}$$

$$\pounds 1776\,..\ 9\,..\ 11\tfrac{3}{4}$$

$$\left.\begin{array}{l}\text{Of the farthing deficient, } \frac{23}{233} \text{ be-}\\ \text{long to H, } \frac{87}{233} \text{ to I, and } \frac{123}{233} \text{ to}\\ \text{K, amounting to - - - -}\end{array}\right\}\text{ - - - - } \frac{1}{4}$$

$$\pounds 1776\,..\ 10\ \text{ Proof.}$$

* This is the Answer given by Dr. Hutton, but is not *strictly* correct.

Quest. 6. $(\pounds 20\times\overset{\text{mo.}}{12})+(\pounds 20\times\overset{\text{mo.}}{8})=$ - - - - - - - - £400 - - - X.

$(\pounds 30\times\overset{\text{mo.}}{12})+(\pounds 20\times\overset{\text{mo.}}{9})+(\pounds 40\times\overset{\text{mo.}}{7})=\pounds 820$ - - - Y.

$(\pounds 60\times\overset{\text{mo.}}{12})+(\pounds 10\times\overset{\text{mo.}}{7})-(\pounds 30\times\overset{\text{mo.}}{6})=\pounds 610$ - - - Z.

$$\text{And}\quad \left.\begin{array}{r}\pounds 400\\ 820\\ 610\end{array}\right\}\text{ are as }\left\{\begin{array}{r}\pounds 40\\ 82\\ 61\end{array}\right.$$

$$\text{Sum }\ \pounds\ 183$$

$$\text{But}\ \left\Vert\begin{array}{lrrrrrrl}
 & & & & & \text{s.} & \text{d.} & \text{qr.}\\
\text{As } \pounds 183 : & \pounds 50 :: & \pounds 40 : & \pounds 10\,.. & 18\,.. & 6\,.. & 3\frac{49}{61} & \text{- - - X.}\\
183 : & 50 :: & 82 : & 22\,.. & 8\,.. & 1\,.. & 0\frac{12}{61} & \text{- - - Y.}\\
183 : & 50 :: & 61 : & 16\,.. & 13\,.. & 4\,.. & 0 & \text{- - - - - Z.}
\end{array}\right.$$

$$\pounds\,50\,..\ 0\,..\ 0$$

SIMPLE INTEREST.

(Page 126.)

Quest. 4. $\dfrac{\pounds 450\times 5}{100}=\dfrac{\pounds 45}{2}=\pounds 22\,..\,\overset{\text{s.}}{10}.$ Ans.

(Key to Vol. I. page 126.)

Quest. 5. $\frac{(\pounds 715 .. \overset{s.}{12} .. \overset{d.}{6}) \times 4\frac{1}{2}}{100} = \pounds 32 .. \overset{s.}{4} .. \overset{d.}{0\frac{3}{4}}$ Ans.

Quest. 6. $\frac{\pounds 720 \times 3}{20} = \pounds 108.$ Ans.

Quest. 7. $\frac{(\pounds 355 .. \overset{s.}{15}) \times 4}{25} = \pounds 56 .. \overset{s.}{18} .. \overset{d.}{4\frac{3}{4}}.$ Ans.

(Page 127.)

Quest. 8. $\frac{(\pounds 32 .. \overset{s.}{5} .. \overset{d.}{8}) \times 4\frac{1}{4} \times 7}{100} = \pounds 9 .. \overset{s.}{12} .. \overset{d.}{1}.$ Ans.

Quest. 9. $\frac{\pounds 170 \times 1\frac{1}{2}}{20} = \pounds 12 .. \overset{s.}{15}.$ Ans.

Quest. 10. $\frac{\pounds 205 .. \overset{s.}{15}}{100} = \pounds 2 .. \overset{s.}{1} .. \overset{d.}{1\frac{3}{4}}.$ Ans.

Quest. 11. $\frac{(\pounds 319 .. \overset{s.}{0} .. \overset{d.}{6}) \times 5\frac{3}{4} \times 3\frac{3}{4}}{100} = \pounds 68 .. \overset{s.}{15} .. \overset{d.}{9\frac{1}{2}}.$ Ans.

Quest. 12. $\frac{\pounds 107 \times 4\frac{3}{4} \times 117}{365 \times 100} = \pounds 1 .. \overset{s.}{12} .. \overset{d.}{7}.$ Ans.

Quest. 13. $\frac{\pounds (17 .. \overset{s.}{5}) \times 4\frac{3}{4} \times 117}{365 \times 100} = \pounds 0 .. \overset{s.}{5} .. \overset{d.}{3}.$ Ans.

(Key to Vol. I. page 127.)

Quest. 14. $7\frac{1}{2}$ per cent. per ann. being at the rate of 5 per cent. for 8 months, *the time given*, it will be

$$\frac{£712\,..\,6)\times 5}{100}\text{, or, }\frac{£712\,..\,\overset{s.}{6}}{20}=£35\,..\,\overset{s.}{12}\,..\,\overset{d.}{3\tfrac{1}{2}}.\quad\text{Ans.}$$

COMPOUND INTEREST.

(Page 128.)

		£	s.	d.	qr.	
Ex. 2.	20)	£50	0	0	0	PRINCIPAL.
		2	10	0	0	First year's interest.
	20)	£52	10	0	0	*Amount in one year.*
		2	12	6	0	Second year's interest.
	20)	£55	2	6	0	*Amount in two years.*
		2	15	1	2	Third year's interest.
	20)	£57	17	7	2	*Amount in three years.*
		2	17	10	$2\frac{3}{10}$	Fourth year's interest.
	20)	£60	15	6	$0\frac{3}{10}$	*Amount in four years.*
		3	0	9	$1\frac{43}{200}$	Fifth year's interest.
	Ans.	£63	16	3	$1\frac{103}{200}$	AMOUNT TOTAL.

Ex. 3. $£1+\dfrac{£1\times\overset{\text{yr.}}{1}}{40}=£1{\cdot}025$ the amount of £1 in $\frac{1}{2}$ year.

And $(£1{\cdot}025)^{10}=1{\cdot}28009$ *Amount of* £1 *in* 10 *half-years.*
by £50 PRINCIPAL.

Prod. $£64\,..\,\overset{s.}{0}\,..\,\overset{d.}{1}$ Amount required.

(Key to Vol. I. page 128.)

Ex. 4. $\pounds 1+\frac{\pounds 1\times\overset{\text{yr.}}{1}}{80}=\pounds 1{\cdot}0125$ the amount of £1 in ¼ year.

And $(\pounds 1{\cdot}0125)^{20}=\pounds 1{\cdot}28202$ *nearly, for 20 quarters.*
by £50 PRINCIPAL.

s. d. qr.
Prod. £64..2..0..0·96 Amount required.

Ex. 5. $\frac{\pounds 1\times 4}{100}+\pounds 1=\pounds 1{\cdot}04$ the amount of £1 at 4 per cent. per ann. for 1 year.

And $(\pounds 1{\cdot}04)^{6}=\pounds 1{\cdot}265319018496$ *Amount of £1 in 6 years.*
by £370 PRINCIPAL.

Prod. £ 468·16803684252 Amount Total.
Subtract 370 PRINCIPAL.

s. d. qr.
Rem. £ 98..3..4..1·3153688192 Int. reqd.

Ex. 6. $\pounds 1+\frac{\pounds 1\times 4\frac{1}{2}}{100}=\pounds 1{\cdot}0225$ the amount of £1 at 4½ per cent. per ann. for ½ year.

And $(\pounds 1{\cdot}0225)^{5}=\pounds 1{\cdot}117676$ *Amount of £1 in 2½ years.*
by £410 PRINCIPAL.

Prod. £458·2476 Amount Total.
Subtract 410 PRINCIPAL.

s. d.
Rem. £ 48..4..11¼ The interest required.

☞ The two last Examples may be more concisely performed without Subtraction, if the *decimal part only*, in the amount of £1 for the time given, be *multiplied* by the PRINCIPAL.

(Key to Vol. I. page 128.)

Ex. 7. 5 per cent. per ann. payable quarterly, being $1\frac{1}{4}$ per cent. per quarter,

			s.	*d.*	qr.	
	80)	£217	0	0	0	PRINCIPAL.
1st qr's int.		2	14	3		
	80)	£219	14	3	0	
2d qr's int.		2	14	11	0·55	
	80)	£222	9	2	0·55	
3d qr's int.		2	15	7	1·506875	
	80)	£225	4	9	2·056875	
4th qr's int.		2	16	3	2·8757109375	
	80)	£228	1	1	0·9325859375	
5th qr's int.		2	17	0	0·66165732421875	
	80)	£230	18	1	1·59424326171875	
6th qr's int.		2	17	8	2·8699280407714843 75	
	80)	£233	15	10	0·46417130249023437 5	
7th qr's. int.		2	18	5	1·5058021412811279296875	
	80)	£236	14	3	1·969973443771362304687 5	
8th qr's int.		2	19	2	0·574624660547142028808593 75	
	80)	£239	13	5	2·5445981043185043334960937 5	
9th qr's int.		2	19	11	0·0818074763039813041687011718 75	
Amount Total.		£242	13	4	2·62640558062248563766479492187 5.	

Otherwise,

The amount of £1 for a quarter of a year, at the rate given, is £1·0125. And

(£1·0125)^9=£1·118292178510262147076129913330078125

multiply by the Principal - - - - - - - - - £217

Amt. Total £242 13 4 2·626405580622485637664794921875.

(Key to Vol. I. page 130.)

ALLIGATION.

ALLIGATION MEDIAL.

lb. s. s. d. s.
Quest. 2. $5 \times 7 = 35 .. 0$ value of Tea at 7.
$9 \times 8\frac{1}{2} = 76 .. 6$ - - - - - - - - 8. d.
$14\frac{1}{2} \times 5\frac{5}{6} = 84 .. 7$ - - - - - - - - 5 .. 10.
s. d.
Divide by $28\frac{1}{2}$ - - -) 196 .. 1 (6 .. $10\frac{1}{2}$ *nearly*, per lb.

gal. s. d. s. d. s. d.
Quest. 3. $4 \times (4 .. 10) = 19 .. 4$ value of Wine at 4 .. 10.
$7 \times (5 .. 3) = 36 .. 9$ - - - - - - - - - 5 .. 3.
$9\frac{3}{4} \times (5 .. 8) = 55 .. 3$ - - - - - - - - - 5 .. 8.
s. d.
Divide by $20\frac{3}{4}$ - - - - - -) 111 .. 4 (5 .. $4\frac{1}{2}$ Ans.

s. d. bu. s. d. s. d.
Quest. 4. $(3 .. 5) \times 3 = 10 .. 3$ value of Flour at 3 .. 5.
$(5 .. 6) \times 4 = 22 .. 0$ - - - - - - - - - 5 .. 6.
$(4 .. 8) \times 5 = 23 .. 4$ - - - - - - - - - 4 .. 8.
s. d. qr.
Divide by 12) 55 .. 7 (4 .. 7 .. $2\frac{1}{3}$ Ans.

Quest. 5. $\dfrac{\overset{\text{s.}}{5} \times \overset{\text{bu.}}{10} + \overset{\text{s.}}{3} \times \overset{\text{bu.}}{18} + \overset{\text{s.}}{2} \times \overset{\text{bu.}}{20}}{48 \text{ bu.}} = \overset{\text{s.}}{3}.$ Ans.

Quest. 6. $\dfrac{\overset{\text{car.}}{22} \times \overset{\text{oz.}}{7} + \overset{\text{car.}}{21} \times \overset{\text{oz.}}{12\frac{1}{2}} + \overset{\text{car.}}{19} \times \overset{\text{oz.}}{17}}{36\frac{1}{2} \text{ oz.}} = 20\frac{19}{73}$ *carats fine*. Ans.

Quest. 7. $\dfrac{\overset{\text{lb.}}{3} \times \overset{\text{oz.}}{9} + (\overset{\text{lb.}}{5} .. \overset{\text{oz.}}{8}) \times \overset{\text{oz.}}{10}}{\overset{\text{lb.}}{3} + \overset{\text{lb.}}{5} .. \overset{\text{oz.}}{8} + \overset{\text{lb.}}{1} .. \overset{\text{oz.}}{10}} = 7\frac{61}{63}$ *ounces fine*. Ans.

(Key to Vol. I. page 132.)

ALLIGATION ALTERNATE.

RULE I.

Quest. 2. $5\left\{\begin{array}{c}6\\4\end{array}\right)$ equal quantities. Ans.

Quest. 3.

$$\begin{array}{c}d.\\7\end{array}\left\{\begin{array}{r}\textit{d.}\\4\\6\\11\end{array}\right)\begin{array}{l}\text{lb.}\\--4\\--4\\--3+1=4\end{array}\ \text{lb.}\left.\begin{array}{c}\\ \\ \\ \end{array}\right\}\text{Therefore equal quantities.}$$

Quest. 4.

$$\begin{array}{c}d.\\46\end{array}\left\{\begin{array}{r}\textit{d.}\\30\\44\\48\\56\end{array}\right)\left.\begin{array}{l}\text{bu.}\quad s.\quad d.\\--\ 2\text{ at }2\,..\,6\\--10\text{ at }3\,..\,8\\--16\text{ at }4\,..\,0\\--\ 2\text{ at }4\,..\,8\end{array}\right\}\text{ or, }\begin{array}{c}d.\\46\end{array}\left\{\begin{array}{r}\textit{d.}\\30\\44\\48\\56\end{array}\right)\left.\begin{array}{l}\text{bu.}\quad s.\quad d.\\-10\text{ at }2\,..\,6\\-\ 2\text{ at }3\,..\,8\\-\ 2\text{ at }4\,..\,0\\-16\text{ at }4\,..\,8\end{array}\right\}$$

Or,

$$\begin{array}{c}d.\\46\end{array}\left\{\begin{array}{r}\textit{d.}\\30\\44\\48\\56\end{array}\right)\left.\begin{array}{l}\qquad\text{bu.}\\--\ 2+10=12\\--10+\ 2=12\\--16+\ 2=18\\--\ 2+16=18\end{array}\right\}\begin{array}{c}\text{which}\\\text{are as}\end{array}\left\{\begin{array}{l}\text{bu.}\quad s.\quad d.\\2\text{ at }2\,..\,6\\2\text{ at }3\,..\,8\\3\text{ at }4\,..\,0\\3\text{ at }4\,..\,8\end{array}\right\}$$

Quest. 5.

$$\begin{array}{c}\text{c. fine}\\21\end{array}\left\{\begin{array}{r}\text{c. fine}\\16\\18\\23\\24\end{array}\right)\left.\begin{array}{l}\text{parts}\quad\text{c. fine.}\\--3\ \text{ of }\ 16\\--2\ \text{ of }\ 18\\--3\ \text{ of }\ 23\\--5\ \text{ of }\ 24\ \text{ or pure.}\end{array}\right\}$$

Quest. 6.

$$\begin{array}{c}s.\\8\end{array}\left\{\begin{array}{r}s.\\12\\10\\1\\0\end{array}\right)\left.\begin{array}{l}\text{gallons.}\\---8----\text{Brandy}\\---7----\text{Wine}\\---2----\text{Cider}\\---4----\text{Water}\end{array}\right\}$$

(Key to Vol. I. page 133.)

RULE II.

Quest. 2.

d.	d.	gal. gal. gal.	gal. gal. gal. gal. s. d.	
64	48	-- 8+ 2=10	As 60 : 10 : : 18 : 3 at 4 .. 0	Ans.
	60	-- 2+ 8=10	60 : 10 : : 18 : 3 at 5 .. 0	
	66	-- 4+16=20	60 : 20 : : 18 : 6 at 5 .. 6	
	72	-- 16+ 4=20	60 : 20 : : 18 : 6 at 6 .. 0	
		Sum 60	18	

RULE III.

(Page 134.)

Quest. 2.

s.	s.	lb.	lb. lb. lb. lb. s.	
8	12	-- 4	As 4 : 20 : : 4 : 20 at 12	Ans.
	10	-- 2	4 : 20 : : 2 : 10 at 10	
	6	-- 2	4 : 20 : : 2 : 10 at 6	
	4	-- 4	4 : 20 : : 4 : 20 at 4	

Quest. 3.

c. fine	c. fine	oz.	oz. oz. oz. oz. c. fine.	
20	15	--------- 2	As 2 : 5 : : 2 : 5 of 15	Ans.
	17	--------- 2	2 : 5 : : 2 : 5 of 17	
	22	5+3+2=10	2 : 5 : : 10 : 25 of 22	
	18	--------- 2	2 : 5 : : 2 : 5 of 18	

SINGLE POSITION.

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Quest. 2. Suppose 12.
$\frac{12 \times 7}{6} = 14$. And, As 14 : 12 : : 21 : 18. Ans.

Quest. 3. Suppose 24.
$24+12+8+6=50$. And, As 50 : 24 : : 75 : 36. Ans.

(Key to Vol. I. page 136.)

Quest. 4. Suppose 5400.
$5400 - 2700 - 1800 = 900$. And, As $9 : 54 : : 1000 : 6000$. Ans.

Quest. 5.

$$\begin{array}{lrcrcr}
 & & & d. & & d. \\
\text{Suppose 4 Men - - - - - -} & 4 & \times & 6 & = & 24 \\
\text{8 Women - - -} & 8 & \times & 4 & = & 32 \\
\text{24 Children - -} & 24 & \times & 2 & = & 48 \\ \cline{6-6}
 & & & \text{Sum} & & 104 \\ \cline{6-6}
\end{array}$$

$$\text{And, As } \overset{d.}{104} : \overset{d.}{52} : : \overset{men.}{4} : \overset{men.}{2} \qquad \text{Hence} \left\{ \begin{array}{l} \text{4 Women} \\ \text{12 Children} \end{array} \right\} \text{Ans.}$$

Quest. 6. Suppose 60 years.

$$\tfrac{3}{5} \text{ of } \overset{yrs.}{60} = \overset{yrs.}{36}, \text{ and } (\overset{yrs.}{36} \times 7) + \overset{yrs.}{40} = 292 \text{ years.}$$

$$\text{But, As } \overset{yrs.}{292} : \overset{yrs.}{219} : : \overset{yrs.}{60} : \overset{yrs.}{45}. \quad \text{Ans.}$$

DOUBLE POSITION.

(Page 139.)

Quest. 2.

$$\begin{array}{llcr}
 & \text{yrs.} & & \text{yrs.} \\
\text{1st. Suppose} & 18 & \text{and} & 54 \\
\text{Subtract} & \text{5 from each} & & 5 \\ \hline
 & 13 & \text{and} & 49 \\ \hline
\end{array}
\qquad
\begin{array}{llcr}
 & \text{yrs.} & & \text{yrs.} \\
\text{2dly. Suppose} & 21 & \text{and} & 63 \\
\text{Subtract} & \text{5 from each} & & 5 \\ \hline
\textit{(5 years before)} & 16 & \text{and} & 58 \\ \hline
\end{array}$$

$$\begin{array}{lrl}
\text{Son } \overset{yrs.}{13} \times 4 = & \overset{yrs.}{52} & \text{Father.} \\
\text{Subtract} & 49 & \text{First result.} \\ \cline{2-2}
\text{1st error} & 3 & \textit{too many.} \\ \cline{2-2}
\end{array}
\qquad
\begin{array}{lrl}
\text{Son } \overset{yrs.}{16} \times 4 = & \overset{yrs.}{64} & \text{Father.} \\
\text{Subtract} & 58 & \text{Second result.} \\ \cline{2-2}
\text{2d error} & 6 & \textit{too many.} \\
\text{Subtract 1st error} & 3 & \\ \cline{2-2}
\textit{Difference} \text{ of the errors} & 3 & \\ \cline{2-2}
\end{array}$$

(Key to Vol. I. page 139.)

21 First supposition.
18 Second supposition.

3 *diff.* of the assumed numbers.
Multiply by 6 greater error.

Divide by the *diff.* of the errors. 3)18(6 to be taken from the greater supposition.

yrs. yrs.
Wherefore 15 and 45 Ans.

work da. id. da. work da. id. da.
Quest. 3. 1st. Suppose 15 and 5 2dly. Suppose 18 and 2
3 1 3 1

45*s.* − 5*s.* = 40*s.* 54*s.* − 2*s.* = 52 *s.*

s. s.
With £2 .. 4 given. With £2 .. 4 given.
Compare 2 .. 0 First result. Compare 2 .. 12 Second result.

4 *too little.* 8 *too much.*

da. da. da.
Now 18 − 15 = 3 *diff.* of the assumed numbers; but

$$\frac{3\times8}{12}=2 \text{ da. to be taken from } 18 \text{ da.}$$
Sum of the errors

Consequently, 16 the number of working days } Ans.
and, 4 the number of idle days. }

gs. gs.
Quest. 4. 1st. Suppose A. 70 and B. 70
won 20 lost 20

End of the 1st Game 90 50
$\frac{2}{3}$ of 90 lost. 60 won 60

End of their Play 30 110

(Key to Vol. I. page 139.)

2dly. Suppose	A. gs. 130	and	B. gs. 130
	won 20		lost 20
End of the 1st Game.	150		110
$\frac{2}{3}$ of 150 lost.	100	won	100
End of their Play.	50		210

With $\overset{gs.}{30} \times 4 = \overset{gs.}{120}$
Compare 110 First result.
10 *too few.*

With $\overset{gs.}{50} \times 4 = \overset{gs.}{200}$
Compare 210 Second result.
10 *too many.*

From which it is evident, that *half the Sum* of the two Suppositions, or, $\frac{\overset{gs.}{70}+\overset{gs.}{130}}{2} = \overset{gs.}{100}$. Ans.

Quest. 5. 1st. Suppose £100 the income.
1st error is £ 20 *too much.*

2dly. Suppose £120 the income.
2nd error is £4 *too much.*

Difference of the errors is 16.

£ 120	second supposition.	£100	first supposition.
20	first error.	4	second error.
£2400	*Prod.*	£400	*Prod.*

And, $\frac{£2400 + 400}{\textit{Diff.}\text{ of the err. } 16} = £125$, the INCOME required.

Lastly, $£125 - \frac{£125}{5} = £100$ per ann. expended by A. and $£125 + \frac{£125}{5} = £150$ per ann. expended by B.

(Key to Vol. I. page 140.)

PRACTICAL QUESTIONS IN ARITHMETIC.

Quest. 1. $\dfrac{\overset{\text{miles}}{100000000}\times\overset{\text{feet}}{5280}}{2000\text{ feet}}=264000000''$ the time of flight.

And $\overset{\text{days}}{365}\;..\;\overset{\text{hrs.}}{6}=31557600''$, wherefore

$$\frac{264000000''}{31557600}=8\tfrac{4808}{13149}\text{ years.}\quad\text{Ans.}$$

Quest. 2. $\dfrac{\overset{\text{miles}}{100000000}\times\overset{\text{feet}}{5280}}{60''\times1500\text{ feet}}=586666\acute{6}\cdot\dot{6}$ time of the ball.

Hence $782222\tfrac{2}{9}$ $\left(=\dfrac{586666\acute{6}\cdot\dot{6}}{7\frac{1}{2}}\right)$ to 1, ratio required.

Quest. 3. $\dfrac{\overset{\text{pa.}}{70}\times\overset{\text{in.}}{28}\times60'}{5280\mathit{ft.}\times12\,\mathit{in.}}=1\tfrac{113}{132}$ mile. Ans.

Quest. 4. $\dfrac{\overset{\text{pa.}}{120}\times\overset{\text{in.}}{28}\times60'}{5280\mathit{ft.}\times12\,\mathit{in.}}=3\tfrac{2}{11}$ miles per hour. Ans.

And $\overset{\text{miles}}{20}\div\overset{\text{miles}}{3\frac{1}{2}}=6\tfrac{2}{7}$ the hours necessary for march.

Also $\overset{\text{hrs.}}{6\frac{2}{7}}+\overset{\text{hr.}}{1}=\overset{\text{hrs.}}{7}\;..\;\overset{\text{min.}}{17\frac{1}{7}}$ Ans.

Quest. 5. $\overset{\text{yds.}}{700}-\overset{\text{yds.}}{220}=\overset{\text{yds.}}{480}$. And $\overset{\text{da.}}{29}-\overset{\text{da.}}{11}=\overset{\text{days}}{18}$.

Now, As $\overset{\text{yds.}}{220}:\overset{\text{yds.}}{480}::\overset{\text{men.}}{12}\times\overset{\text{da.}}{11}=132:288$, *men involved in days.*

But $\dfrac{288}{\text{days }18}=16$ men, the numb. that must work.

Lastly, $\overset{\text{men}}{16}-\overset{\text{men}}{12}=\overset{\text{men}}{4}$ additional required.

Quest. 6. $\dfrac{\overset{\text{inches.}}{500000000}}{12\times3\times5\frac{1}{2}\times40\times8}=\overset{\text{miles.}}{7891}\;..\;\overset{\text{yds.}}{728}\;..\;\overset{\text{ft.}}{2}\;..\;\overset{\text{in.}}{8}$. Ans.

(Key to Vol. I. page 140.)

Quest. 7. $\overset{\text{yds.}}{11} \times 3' = \overset{\text{yds.}}{33}$ travelled by A in 3′, that is, every 34 yards of B gain 1 yard on A; now half the circuit of the wood=268 yards, therefore

As $\overset{\text{yd.}}{1} : \overset{\text{yds.}}{34} :: \overset{\text{yds.}}{268} : \overset{\text{yds.}}{9112}$ travelled by B when he overtakes A. But $\frac{\overset{\text{yds.}}{9112}}{268} = 17$ revolutions made by B, and consequently $16\frac{1}{2}$ by A.

(Page 141.)

Quest. 8. It is manifest that $\frac{1}{12}$ of the work=A's daily labour; and that $\frac{1}{14}$ of it=the daily labour of B.

But $\frac{1}{12}+\frac{1}{14}=\frac{26}{168}$ parts of the work performed in a day when both work.

Again, $\frac{26}{168})1(6\frac{6}{13}$ *days.* Ans.

Or, more clearly, As $\frac{26}{168}$, that is, $\frac{13}{84}$: whole work=Unity or 1 :: 1 day : $6\frac{6}{13}$ days.

Quest 9. $\frac{3}{4}$ of $\frac{3}{5}=\frac{9}{20}$.

And, $\frac{9}{20}) \overset{\pounds}{1800} (\pounds 4000.$ Ans.

Quest. 10.

1st. Suppose £100		2dly. Suppose £400
£25+£20= 45		£100+£20= 120
£55		£280
80	should be	230
1st error £25 *too little.*		2d error £50 *too much.*

Wherefore, $\frac{\overset{\text{1st sup.}}{\pounds 100} \times \overset{\text{2d err.}}{50} + \overset{\text{2d sup.}}{\pounds 400} \times \overset{\text{1st error.}}{25}}{\textit{Sum}\text{ of the errors, } 25+50} = \pounds 200.$ Ans.

Quest. 11. Because the hands appear in conjunction 11 times in 12 hours, and that their respective velocities are always in the same ratio, it follows that,

As $11 : 12 :: \overset{\text{hr.}}{1} : \overset{\text{hr.}}{1\frac{1}{11}} = \overset{\text{hr.}}{1} .. \overset{\text{min.}}{5} .. \overset{\text{sec.}}{27\frac{3}{11}}$. Ans.

(Key to Vol. I. page 141.)

Quest. 12. $£21 \times \overset{\text{we.}}{52} = £1092$.
And $£1500 - £1092 = £408$, *saved.* Ans.

Quest. 13.

$$\begin{array}{rcl} \overset{\text{oranges.}}{180} \div 2 & = & \overset{d.}{90} \\ 180 \div 3 & = & 60 \\ \hline 5)\,\underline{360} & & 150 \text{ prime cost.} \\ 72 \times 2 & = & 144 \text{ sold for.} \\ \hline & & 6 \text{ pence } \textit{loss.} \quad \text{Ans.} \end{array}$$

Quest. 14. As $\overset{\text{we.}}{20} \times \overset{\text{oz.}}{8} : \overset{\text{we.}}{12} \times \overset{\text{oz.}}{20} :: \overset{\text{men.}}{1500} : \overset{\text{men.}}{2250}$. Ans.

Quest. 15. $\dfrac{15550 \text{ miles}}{23 \text{hrs. } 56'} = 649\frac{259}{359}$ miles per hour. Ans.

Quest. 16. $\dfrac{1}{4} + \dfrac{3}{7} \text{ of } \dfrac{3}{4} = \dfrac{4}{7}$. And $1 - \dfrac{4}{7} = \dfrac{3}{7}$, *remaining.*

But $\frac{3}{7})£820\,(£1913 \,..\, \overset{s.}{6} \,..\, \overset{d.}{8}$. Ans.

Quest. 17. As $\overset{\text{men}}{1500} \times \overset{\text{we.}}{8} : \overset{\text{men}}{1000} \times \overset{\text{we.}}{5} :: \overset{\text{oz.}}{16} : \overset{\text{oz.}}{6\frac{2}{3}}$. Ans.

Quest. 18. As $7 : 9 :: £8400 : £10800$ *elder brother's* FORTUNE.
And $£10800 + £8400 = £19200$. Ans.

(Page 142.)

Quest. 19. As $\overset{\text{hrs.}}{11} : \overset{\text{hrs.}}{12} :: \overset{\text{hrs.}}{5} : \overset{\text{hrs.}}{5\frac{5}{11}} = \overset{\text{h.}}{5} \,..\, \overset{\text{min.}}{27} \,..\, \overset{\text{sec.}}{16\frac{4}{11}}$. Ans.

Quest. 20. $3 \times 5 \times \overset{\text{men}}{20} = \overset{\text{men}}{300}$. Ans.

For it is evident that, if the work be three times as great, and the time but one fifth part the former, fifteen times as many men will be required.

(Key to Vol. I. page 142.)

Quest. 21. $1-\frac{7}{18}=\frac{11}{18}$ the residue after the elder son's fortune.

And $\frac{7}{18}$ of $\frac{11}{18}=\frac{77}{324}$ the younger son's patrimony.

Now $\frac{11}{18}=\frac{126}{324}$, and $\frac{126-77}{324}=\frac{49}{324}$ the difference of the son's portions. $=£514\frac{1}{3}$ *by the question.*

Also $\frac{126+77}{324}=\frac{203}{324}$ the sum of both sons' property; therefore $1-\frac{203}{324}=\frac{121}{324}$ the provision made for the RELICT.

And As $\overset{\text{shares}}{49} : \overset{\text{shares}}{121} :: £514\frac{1}{3} : £1270 \,..\, \overset{s.}{1} \,..\, \overset{d.}{9\frac{11}{49}}$. Ans.

Quest. 22. $1-\frac{13}{20}=\frac{7}{20}$ of the estate, for the *younger* child.

But $\frac{13-7}{20}=\frac{6}{20}=\frac{3}{10}$, therefore,

As 3 : 10 :: £1200 : £4000. Ans.

Quest. 23. $\overset{\text{miles}}{1\frac{1}{2}} \times \overset{\text{hrs.}}{7} = \overset{\text{miles}}{10\frac{1}{2}}$ A. rode more than B. And

$\frac{\overset{\text{miles}}{10\frac{1}{2}}}{2} = \overset{\text{miles.}}{5\frac{1}{4}}$. Also $\frac{\overset{\text{miles}}{100}}{2} = \overset{\text{miles}}{50}$.

But $\overset{\text{miles}}{50} + \overset{\text{miles}}{5\frac{1}{4}} = \overset{\text{miles}}{55\frac{1}{4}}$ for A. And

$50 - 5\frac{1}{4} = 44\frac{3}{4}$ for B.

Lastly $\frac{\overset{\text{miles}}{55\frac{1}{4}}}{7} = \overset{\text{miles}}{7\frac{25}{28}}$ = A's rate per hour.

And $\frac{\overset{\text{miles}}{44\frac{3}{4}}}{7} = \overset{\text{miles}}{6\frac{11}{28}}$ B's rate per hour.

} Ans.

(Key to Vol. I. page 142.)

Quest. 24. From $\overset{\text{h.}}{8}$. A. M. to $\overset{\text{h.}}{4}$. P. M. there are 8 hours, which $\times^{\text{ed}}$ by 3, give 24 miles walked by A, when B sets out from London.

Also $\overset{\text{miles}}{130} - \overset{\text{miles}}{24} = \overset{\text{miles}}{106}$; And $\overset{\text{miles}}{4} + \overset{\text{miles}}{3} = \overset{\text{miles}}{7}$ per hour.

But $\dfrac{\overset{\text{miles}}{106}}{7} = 15\frac{1}{7}$ hours, the time of B. on the road.

Again $(\overset{\text{hrs.}}{15\frac{1}{7}} \times \overset{\text{miles}}{3}) + \overset{\text{miles}}{24} = \overset{\text{miles}}{69\frac{3}{7}}$, A's distance from Exeter when he meets B, consequently $60\frac{4}{7}$ miles from London.

Quest. 25. $(\overset{\text{yds.}}{100} - \overset{\text{yd.}}{1}) \times 50 \times 2 = \overset{\text{yds.}}{10100} = \overset{\text{miles}}{5} \,..\, \overset{\text{yds.}}{1300}$. Ans.

Quest. 26. $(\overset{\text{str.}}{24} + \overset{\text{str.}}{1}) \times \overset{\text{hrs.}}{12} = \overset{\text{str.}}{300}$. Ans.*

* See this Question, page 113, line 1, vol. i. of Hutton's Course.

Quest. 27. $\dfrac{(1 \times 2^{63} \times 2) - 1}{(2-1) \times 7680 \times 512 \textit{ pints in } 1 \textit{ qr.}} = \overset{\text{quarters}}{4691249611844} \,..\, \overset{\text{bu.}}{2} \,..\, \overset{\text{p.}}{0} \,..\, \overset{\text{gal.}}{1}$

And $\overset{\text{qrs.}}{4691249611844} \,..\, \overset{\text{bu.}}{2} \,..\, \overset{\text{p.}}{0} \,..\, \overset{\text{gal.}}{1} \times \overset{s.}{27} \,..\, \overset{d.}{6} =$
£$6450468216285 \,..\, \overset{s.}{17} \,..\, \overset{d.}{3} \,..\, \overset{\text{qr.}}{3\frac{32757}{32768}}$. Ans.

(Page 143.)

Quest. 28.

1st. Suppose	£400	2dly. Suppose	£464
End of 1st year it is	600	End of 1st year it is	680
2d year - - -	850	2d year - - -	950
3d year - -	1162·5	3d year - -	1287·5
4th year -	1553·125	4th year - -	1709·375
	10342·1875	should be	10342·1875

1st error £ 8789·0625 *too little.* 2d err. £8632·8125 *too little.*

The *difference* of the errors is £156·25

The *difference* of the products arising is £625000.

But $\dfrac{£625000}{£\ 156{\cdot}25} = £4000$. Ans.

(Key to Vol. I. page 143.)

Quest. 29. $(£1012 \,..\, \overset{s.}{10}) - £750 = £262 \,..\, \overset{s.}{10}$ the amount of *int.*

$$\text{And } \frac{£262 \,..\, \overset{s.}{10}}{7 \text{ yrs.}} = £37 \,..\, 10 \text{ the } \textit{yearly} \text{ interest.}$$

$$\text{Lastly } \frac{(£37 \,..\, \overset{s.}{10}) \times \overset{£}{100}}{£750} = £5 \text{ per cent. Ans.}$$

Quest. 30.

1st. Suppose	A.	£400	2dly. Suppose	A.	£600
P. Quest.	B.	185	*P. Quest.*	B.	385
	C.	280		C.	480
		£865			£1465
		1000	should be		1000
1st error		£135 *too little.*	2d error		£465 *too much.*

$$\text{And } \frac{£135 \times £600 + £465 \times £400}{£135 + £465} = £445 = \text{A's dividend.}$$

Wherefore £445 for A. £230 for B. and £325 for C. Ans.

Quest. 31. $4+5=9$, and $\frac{\overset{\text{hrs.}}{12}}{9} = \overset{\text{h.}}{1\frac{1}{3}}$ one of the fifths of the time till midnight.

But $\overset{\text{hr.}}{1\frac{1}{3}} \times 4 = \overset{\text{hrs.}}{5\frac{1}{3}} = \overset{\text{hrs.}}{5} \,..\, \overset{\text{min.}}{20}$, consequently 20 *min. past* 5. Ans.

Quest. 32. $\frac{2}{5}$ of $\frac{4}{9}$ of $\frac{3}{16} = \frac{1}{30} = \frac{8}{240}$, and $\frac{3}{16} = \frac{45}{240}$.

But $\frac{45-8}{240} = \frac{37}{240}$ left unsold.

Also $£1200 \times \frac{37}{240} = £185$ the worth of $\frac{37}{240}$ of the ship. Ans.

(Key to Vol. I. page 143.)

Quest. 33.

$$\begin{array}{rrlcrrl} & & \text{ac.} & & & & \text{ac.} \\ \text{1st. Suppose} & \text{A.} & 200 & \qquad & \text{2dly. Suppose} & \text{A.} & 400 \\ \textit{Per Quest.}\left\{\begin{array}{l}\\ \\ \end{array}\right. & \begin{array}{r}\text{B.}\\ \text{C.}\end{array} & \begin{array}{r}300\\ 364\end{array} & & \textit{Per Quest.}\left\{\begin{array}{l}\\ \\ \end{array}\right. & \begin{array}{r}\text{B.}\\ \text{C.}\end{array} & \begin{array}{r}500\\ 564\end{array} \\ \hline & & 864 & & & & 1464 \\ & & 1200 & \text{should be} & & & 1200 \\ \hline & \text{1st error} & 336\ \textit{too little.} & & & \text{2d error} & 264\ \textit{too much.} \end{array}$$

$$\text{And}\ \frac{\overset{\text{ac.}}{336}\times\overset{\text{ac.}}{400}+\overset{\text{ac.}}{264}\times\overset{\text{ac.}}{200}}{\overset{\text{ac.}}{264}+\overset{\text{ac.}}{336}}=\overset{\text{ac.}}{312}.\quad \text{A's share.}$$

Wherefore 312 acres for A. 412 acres for B. and 476 acres for C. Ans.

Quest. 34. $\frac{2}{7}$ of $\frac{3}{8}=\frac{9}{84}$. And $\frac{8}{15}$ of $\frac{5}{16}=\frac{14}{84}$.

But $10\frac{9}{84}-\frac{14}{84}=9\frac{79}{84}$. Ans.

Quest. 35. $\frac{2}{3}$ of $\frac{4}{5}$ of $1\frac{1}{2}=\frac{4}{5}$. Also $1\div\frac{4}{5}=\frac{5}{4}$.

And $\frac{5}{4}\times\frac{5}{4}=\frac{25}{16}=1\frac{9}{16}$. Ans.

Quest. 36. As $\overset{\text{in.}}{8\frac{1}{2}} : \overset{\text{in.}}{12} :: \overset{\text{in.}}{12} : \overset{\text{in.}}{16\frac{16}{17}}$. Ans.

Quest. 37. As $\overset{\text{mo.}}{12} : \overset{\text{mo.}}{15}$ :: £5 : £$6\frac{1}{4}$.

And, As £100 + £$6\frac{1}{4}$: £100 :: £$138\frac{1}{8}$: £130. Ans.

Quest. 38.

$$\left.\begin{array}{lll} \text{A. £4.} & \text{B. £3} & \\ \text{A.}\ \ 8. & \text{B.}\ \ 6 & \text{C. £5} \end{array}\right\}\ \textit{Per Quest.}$$

But £8 + £6 + £5 = £19, therefore

$\frac{8}{19}$, $\frac{6}{19}$, and $\frac{5}{19}$, of the fortune, are the three shares.

Now $\frac{8}{19}$) £4000 *A's share* (£9500. Ans.

Or, more clearly, As 8 : 19 :: £4000 : £9500 the whole FORTUNE.

(Key to Vol. I. page 144.)

Quest. 39. As $60' : \frac{2}{3}' :: \overset{\text{yds.}}{1760} \times 10 : \frac{1}{9}$ of $\overset{\text{yds.}}{1760} = \overset{\text{yds.}}{195\frac{5}{9}}$ the hare had increased her distance from the dog before she was observed.

And $\overset{\text{miles}}{18} - \overset{\text{miles}}{10} = \overset{\text{miles}}{8}$ per hour, the excess of the dog's speed above that of the hare. Wherefore,

As $\overset{\text{yds.}}{1760} \times 8 : \overset{\text{yds.}}{40} + \overset{\text{yds.}}{195\frac{5}{9}} :: 60' : 1' .. 0''\frac{5}{22}$ *time of the course.*

Also As $60' : 1' .. 0''\frac{5}{22} = 1'\frac{1}{264} :: \overset{\text{yds.}}{1760} : \overset{\text{yds.}}{530}$ run by the *Dog.*

Quest. 40. Prudent Officer's annual state.

Divide by 20 and add, *yearly.*

£.	s.	d.	qr.	years.
10	0	0		1st.
20	10	0		2d.
31	10	6		3d.
43	2	0	1,2	4th.
55	5	1	2,06	5th.
68	0	4	2,36	6th.
81	8	4	3,28	7th.
95	9	9	3,44	8th.
110	5	3	3,01	9th.
125	15	6	3,76	10th.
142	1	4	1,15	11th.
159	3	5	0,41	12th.
177	2	7	0,63	13th.
195	19	8	2,86	14th.
215	15	8	2,21	15th.
236	11	5	3,92	16th.
258	8	0	3,51	17th.
281	6	5	2,89	18th.
305	7	9	0,43	19th.
330	13	1	3,05	20th.
357	3	9	2,61	21st.
385	0	11	3,74	22d.
414	6	0	2,13	23d.
445	0	4	0,43	24th.
477	5	4	1,25	25th.
511	2	7	2,12	26th.
546	13	9	0,42	27th.
584	0	5	1,44	28th.
623	4	6	2,52	29th.
664	7	9	1,44	30th.
707	12	2	0,31	31st.
752	19	9	1,53	32d.

	s.	d.	qr.	
£400	0	0	0	Annuity
37	12	11	3,47	=int. of 752 19 9 1,53
Sum £437	12	11	3,47	to spend per ann. Ans.

(Key to Vol. I. page 144.)

Prodigal Officer's annual state.

	£.	s.	d.	qr.	years.
£10+£1..10+£0..13..9..2,4 *insur.* =	12	3	9	2,4	1st.
£0..12..2..1,12+£11..10+*ins.* £1..9..1..3,65 =	25	15	1	3,17	2d.
(In which manner proceed *annually*.)	40	17	1	3,74	3d.
	57	13	3	2,41	4th.
	76	7	5	0,28	5th.
	97	3	9	3,42	6th.
	120	7	3	1,11	7th.
	146	3	1	0,56	8th.
	174	17	2	1,82	9th.
	206	16	2	1,06	10th.
	242	7	4	2,15	11th.
	281	18	11	0,73	12th.
	325	19	11	0,69	13th.
	375	0	5	3,20	14th.
	429	11	10	0,22	15th.
	490	6	6	0,75	16th.
	557	18	5	1,35	17th.
	633	3	1	3,11	18th.
	716	17	10	1,90	19th.
	810	1	10	0,92	20th.
	913	16	5	2,17	21st.
	1029	5	10	3,30	22d.
	1157	15	10	3,57	23d.
	1300	16	3	3,61	24th.
	1459	19	11	2,56	25th.
	1637	3	4	1,60	26th.
	1834	7	2	0,02	27th.
	2053	16	7	1,60	28th.
	2168	0	4	3,88	29th.
s. *d.* qr.	2424	10	1	2,78	30th.
£400..0..0..0 Annuity	2709	19	7	1,09	31st.
333..0..11..1,53=11 per ct. on	3027	14	0	3,76	32d.

Diff. £ 66..19..0..2,46 to spend per ann. Ans.

END OF THE ARITHMETIC.

LOGARITHMS.

(Key to Vol. I. page 160.)

	Numbers.	Logarithms.
Ex. 7.	3·1416	0·497151
	82·	1·913814
	73 its log. ·863323 41 its log. ·612784 diff.	0·250539
	Prod. 458·6736	2·661504

	Numbers.	Logarithms.
Ex. 8.	·02196	−2·4647875
	751·3	2·8758134
	6	0·7781513
	941 log. arithmetical comp.	−3·0264104
	Prod. ·1396891	−1·1451626

	Numbers.	Logarithms.
Ex. 9.	As 7241 log. arithmetical comp.	−4·1402015
	: 3·58	0·5538830
	: : 20·46	1·3109056
	: ·01011556	−2·0049901

	Numbers.	Logarithms.
Ex. 10	As $\sqrt{724}$. log. arithmetical comp.	−2·5701307
	: $\sqrt{\frac{58}{13}}$.	0·3247423
	: : 6·927.	0·8405452
	: ·54377375	−1·7354182

ALGEBRA.

ADDITION OF ALGEBRA.

(Key to Vol. I. page 167.)

CASE I.

Ex. 7. Ans. $69xy$. Ex. 8. Ans. $-29y^2$.

Ex. 9. Ans. $28a-20b$. Ex. 10. Ans. $93-73\sqrt{x}-20xy$.

Ex. 11. Ans. $21xy-15x+20ab$.

(Page 168.)

CASE II.

Ex. 4. Ans. $6a^2$. Ex. 5. Ans. $-19b^2y^3$.

Ex. 6. Ans. $3ab-5$. Ex. 7. Ans. $3\sqrt{a^2x}=3ax^{\frac{1}{2}}$.

Ex. 8. Ans. $-\sqrt{ax}$. Ex. 9. Ans. $4y+7ax^{\frac{1}{2}}$.

(Page 169.)

CASE III.

Ex. 4. Ans. $9x^2y^2-11x^2y+3axy$.

Ex. 5. Ans. $25ax+x^2+3xy+8y^2+26$.

Ex. 6. Ans. $9\sqrt{ax}+2x+7\sqrt{xy}+5y+19$.

Ex. 7. Ans. $-3x^2y-3xy^2$.

(Key to Vol. I. page 169.)

Ex. 8. Ans. $5\surd xy+17x+4\surd x-y-9$.
Ex. 9. Ans. $3a^2+3a+x^2-3x+\surd x-2y-4$.
Ex. 10. Ans. $4a-4b$.
Ex. 11. Ans. $8a-12x$.
Ex. 12. Ans. $2x-4a-b+5$.
Ex. 13. Ans. $10b-3a+c$.
Ex. 14. Ans. $2a$.
Ex. 15. Ans. $-2b-3c-d-17$.
Ex. 16. Ans. $b^2+2ab+bc-b-c$.
Ex. 17. Ans. $a^3+ab^2-abc+b^2c$.
Ex. 18. Ans. $12x+6a-4b-12c+40$.

SUBTRACTION OF ALGEBRA.

(Page 170.)

Ex. 7. Ans. $10x^2y+4$.
Ex. 8. Ans. $-2\surd xy+2x\surd xy+2xy-3$.
Ex. 9. Ans. $-2x^2+\surd x-2b-6$.

(Page 171.)

Ex. 10. Ans. $-2xy+20$.
Ex. 11. Ans. $7x^3-2x^2+2a+2b$.
Ex. 12. Ans. $3xy^3-4x^2y^2+8a(xy+10)^{\frac{1}{2}}$.
Ex. 13. Ans. $2b$.
Ex. 14. Ans. $3a+3b$.
Ex. 15. Ans. $a-9b$.

(Key to Vol. I. page 171.)

Ex. 16. Ans. $4a-9x$.

Ex. 17. Ans. $-4x-5a+3b-3$.

Ex. 18. Ans. $a+b-10$.

Ex. 19. Ans. $c-d$.

Ex. 20. Ans. $-3a^2+2ab-3c+bc-b$.

Ex. 21. Ans. $a^3+3b^2c-b^2$.

Ex. 22. Ans. $8x+9a-8b-6d+50$.

Ex. 23. Ans. $x-12a-2b+12c-10$.

Ex. 24. Ans. $10x+14a-8b-7c$.

MULTIPLICATION OF ALGEBRA.

(Page 172.)

When both Factors are Simple Quantities.

Ex. 9. Ans. $-12ax^2$. Ex. 10. Ans. $6acx$.

Ex. 11. Ans. $-12xy$. Ex. 12. Ans. $25ax^2yz$.

(Page 173.)

When one of the Factors is a Compound Quantity.

Ex. 4. Ans. $48ax-8a^2c$.

Ex. 5. Ans. $-50ac+14ab$.

Ex. 6. Ans. $8abx-2ab^2+6a^2b^2$.

Ex. 7. Ans. $12c^2xy+4x^2y$.

Ex. 8. Ans. $-40x^4+12x^2y^2$.

Ex. 9. Ans. $6a^3x^2-4ax^4-12abx^2$.

(Key to Vol. I. page 174.)

When both Factors are Compound Quantities.

Ex. 1.
$$\begin{array}{rl} \text{Mul.} & 10ac \\ \text{by} & 2a \\ \hline \textit{Prod.} & 20a^2c \\ \hline \end{array}$$

Ex. 2.
$$\begin{array}{rl} \text{Mul.} & 3a^2-2b \\ \text{by} & 3b \\ \hline \textit{Prod.} & 9a^2b-6b^2 \\ \hline \end{array}$$

Ex. 3.
$$\begin{array}{rl} \text{Mul.} & 3a+2b \\ \text{by} & 3a-2b \\ \hline \textit{Prod.} & 9a^2 \quad * \; -4b^2 \\ \hline \end{array}$$

Ex. 4.
$$\begin{array}{rl} \text{Mul.} & x^2-xy+y^2 \\ \text{by} & x+y \\ \hline \textit{Prod.} & x^3 \quad * \quad * \quad +y^3 \\ \hline \end{array}$$

Ex. 5.
$$\begin{array}{rl} \text{Mul.} & a^3+a^2b+ab^2+b^3 \\ \text{by} & a-b \\ \hline & a^4+a^3b+a^2b^2+ab^3 \text{ - - - - - } \textit{Prod.} \text{ by } a. \\ & \quad -a^3b-a^2b^2-ab^3-b^4 \text{ - - } \textit{Prod.} \text{ by } \quad -b. \\ \hline & a^4 \quad * \quad * \quad * \quad -b^4 \;\; \textit{Product} \text{ by } a-b. \\ \hline \end{array}$$

Ex. 6. Ans. $a^4+a^2b^2+b^4$.

Ex. 7. Ans. $3x^4+4x^3y-13x^2-4x^2y^2+24xy-30$.

Ex. 8. Ans. $9a^4-18a^3x+2a^2x^2-6ax^3-35x^4$

Ex. 9. Ans. $6x^6-5x^5y^2+15x^3y^3-6x^4y^4-3x^2y^5+9y^6$.

Ex. 10. Ans. $a^3-a^2b-ab^2-2b^3$.

DIVISION OF ALGEBRA.

(Page 175.)

When the Divisor and Dividend are both Simple Quantities.

Ex. 3. $8x)16x^2(2x$. Quotient.

Ex. 4. $-3a^2x)12a^2x^2(-4x$. Quotient.

Ex. 5. $3ay)-15ay^2(-5y$. Quotient.

Ex. 6. $-8axz)-18ax^2y(\frac{9xy}{4z}$. Quotient.

(Key to Vol. I. page 176.)

When the Divisor is Simple, and the Dividend a Compound Quantity.

Ex. 4. Ans. $3b-4x+\frac{1}{2}$. Ex. 5. Ans. $x+2+\frac{2a-5}{x}$.

Ex. 6. Ans. $4x-3a+2c$. Ex. 7. Ans. $2a^2-3x-5$.

Ex. 8. Ans. $3x^2-3ab-\frac{d^2}{c}$ Ex. 9. Ans. $\frac{5+y}{7}-\frac{6y}{7a}$.

Ex. 10. Ans. $\frac{3\frac{1}{3}d^2b}{a}-10b^2$. or, $\frac{10d^2b}{3a}-10b^2$.

(Page 178.)

When the Divisor and Dividend are both Compound Quantities.

Ex. 1. $a+2x)a^2+4ax+4x^2(a+2x$. Quotient.

$$\begin{array}{l} a^2+2ax \\ \hline \cdot \quad 2ax+4x^2 \\ \quad\;\; 2ax+4x^2 \\ \hline \quad\;\; \cdot \quad \cdot \end{array}$$

Ex. 2. $a-z)a^3-3a^2z+3az^2-z^3(a^2-2az+z^2$. Quotient.

$$\begin{array}{l} a^3-\;a^2z \\ \hline \cdot -2a^2z+3az^2 \\ \;\; -2a^2z+2az^2 \\ \hline \cdot \qquad az^2-z^3 \\ \qquad\quad az^2-z^3 \\ \hline \qquad\quad \cdot \quad \cdot \end{array}$$

Ex. 3. $1+a)1$ - - - - $(1-a+a^2-a^3+a^4-a^5+a^6-$&c. Quot.

$$\begin{array}{l} 1+a \\ \hline -a \text{ rem.} \\ -a-a^2 \\ \hline +a^2 \text{ rem.} \\ +a^2+a^3 \\ \hline -a^3 \text{ rem. \&c.} \\ \hline \end{array}$$

(Key to Vol. I. page 178.)

Ex. 4.

$$\begin{array}{l}
3x-6)12x^4 \text{ - - - - - - - - - - - - - - - - } 192(4x^3+8x^2+16x+32.\ \text{Quo.} \\
\quad 12x^4-24x^3 \\
\hline
\cdot \quad 24x^3 \\
\quad 24x^3-48x^2 \\
\hline
\cdot \quad 48x^2 \\
\quad 48x^2-96x \\
\hline
\cdot \quad 96x-192 \\
\quad 96x-192 \\
\hline
\cdot \quad \cdot
\end{array}$$

Ex. 5. Divisor $a^2-2ab+b^2)$

Quotient $(a^3-3a^2b+3ab^2-b^3.$

$$\begin{array}{l}
\text{Dividend}\quad a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 \\
\quad a^5-2a^4b+\ a^3b^2 \\
\hline
\cdot \ -3a^4b+\ 9a^3b^2-10a^2b^3 \\
\quad -3a^4b+\ 6a^3b^2-\ 3a^2b^3 \\
\hline
\cdot \quad 3a^3b^2-\ 7a^2b^3+5ab^4 \\
\quad 3a^3b^2-\ 6a^2b^3+3ab^4 \\
\hline
\cdot \quad -\ a^2b^3+2ab^4-b^5 \\
\quad -\ a^2b^3+2ab^4-b^5 \\
\hline
\cdot \quad \cdot \quad \cdot
\end{array}$$

Ex. 6.

$$\begin{array}{l}
2z-3a)48z^3-96az^2-64a^2z+150a^3(24z^2-12az-50a^2.\ \text{Quot.} \\
\quad * \ -24az^2-64a^2z \\
\quad\quad * \ -100a^2z+150a^3 \\
\hline
\cdot \quad \cdot
\end{array}$$

Ex. 7. Divisor $b^3-3b^2x+3bx^2-x^3)$

Quotient $(b^3+3b^2x+3bx^2+x^3$

$$\begin{array}{l}
\text{Dividend }) b^6 \ * \ -3b^4x^2 \ * \ +3b^2x^4 \ * \ -x^6 \\
\quad * \ 3b^5x-6b^4x^2+b^3x^3+3b^2x^4 \\
\quad\quad * \ 3b^4x^2-8b^3x^3+6b^2x^4 \\
\quad\quad\quad * \ b^3x^3-3b^2x^4+3bx^5-x^6 \\
\hline
\cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

(Key to Vol. I. page 178.)

Ex. 8. Divisor $a-x)$

$$)\underset{\cdot}{a^7}\;\;\underset{\cdot}{*}\;\;\underset{\cdot}{*}\;\;\underset{\cdot}{*}\;\;\underset{\cdot}{*}\;\;\underset{\cdot}{*}\;\;\underset{\cdot}{-x^7}\,(a^6+a^5x+a^4x^2+a^3x^3+a^2x^4+ax^5+x^6.$$

Quotient.

Ex. 9. Ans. $a^2+4ax+x^2$.

Ex. 10. Ans. $a^3-2a^2b+8ab^2-16b^3$.

Ex. 11. Ans. $8a^3+5\frac{1}{3}a^2b+3\frac{5}{9}ab^2+2\frac{10}{27}b^3+\frac{3\frac{20}{27}b^4}{3a-2b}$.

ALGEBRAIC FRACTIONS.

(Page 179.)

To reduce a Mixed Quantity to an Improper Fraction.

Ex. 3. $\frac{(5\times 7)+3}{7}=\frac{38}{7}$ Ans.

Ex. 4. $1-\frac{3a}{x}=\frac{x}{x}-\frac{3a}{x}=\frac{x-3a}{x}$ the fraction required.

Ex. 5. $\frac{2a\times 4x}{4x}-\frac{3ax+a^2}{4x}=\frac{8ax-3ax+a^2}{4x}=\frac{5ax+a^2}{4x}$ frac. req.

Ex. 6. Ans. $\frac{64x-18}{5x}$.

Ex. 7. Ans. $\frac{cx+1-3a-c}{c}$.

Ex. 8. Ans. $\frac{20a+10ax-2x^3+3a}{5a}$.

(Key to Vol. I. page 180.)

To reduce an Improper Fraction to a Whole or Mixed Quantity.

Ex. 3. $5)33(6\frac{3}{5}$. And

$$a)2ax-3x^2(2x-\frac{3}{a}x^2.$$

Therefore $6\frac{3}{5}$, and $2x-\frac{3x^2}{a}$ are mixed quantities equivalent to the given fractions.

Ex. 4. Ans. $2ax$, and $2a+2b+\frac{4b^2}{a-b}$.

Ex. 5. Ans. $3x-3y$, and $2x^2+2xy+2y^2$.

Ex. 6. Ans. $2a-\frac{4}{5}+\frac{6}{5a}$. Or, $2a-\frac{4a-6}{5a}$* $=2a+\frac{6-4a}{5a}$.

Ex. 7. Ans. $5-\frac{5a^2+10a+20}{3a^3+2a^2-2a-4}$.

* *Here, if the fractional part* $\frac{4}{5}+\frac{6}{5a}$ *be reduced to a common denominator and expressed as one quantity, the sign* + *of* $\frac{6}{5a}$ *must be changed to* minus, *since the leading sign denotes that the whole compound quantity is to be subtracted, affecting* $\frac{6}{5a}$ *in a manner* contrary *to that intended by* $-\frac{4}{5}+\frac{6}{5a}$. [See an Example, the last line, page 88 of the KEY.] *Now, because changing* plus *to* minus, *and* minus *to* plus, *is in reality changing Addition to Subtraction, and Subtraction to Addition, it follows that, in all compound quantities under the same* VINCULUM, *of which the leading sign is* minus, *all the other signs must invariably be made* contrary *when the* VINCULUM is broken. ☞ *The Course offers no Explanation on this subject !*

(Page 181.)

To reduce Fractions to a Common Denominator.

Ex. 3. $2a\times 2c=4ac$ First Numerator.
$3b\times x=3bx$ Second Numerator.
And $x\times 2c=2cx$ Common Denominator.

Therefore $\frac{4ac}{2cx}$, and $\frac{3bx}{2cx}$. Ans.

(Key to Vol. I. page 181.)

Ex. 4. $2a \times 2c = 4ac$ First Numerator.
$(3a+2b) \times b = 3ab + 2b^2$ Second Numerator.
And $b \times 2c = 2bc$ Common Denominator.

Wherefore $\frac{4ac}{2bc}$, and $\frac{3ab+2b^2}{2bc}$ are the fractions required.

Ex. 5. $5a \times 2c \times 1 = 10ac$ First Numerator.
$3b \times 3x \times 1 = 9bx$ Second Numerator.
$4d \times 3x \times 2c = 24cdx$ Third Numerator.
And $3x \times 2c \times 1 = 6cx$ Common Denominator.

Consequently the new fractions are $\frac{10ac}{6cx}$, $\frac{9bx}{6cx}$, and $\frac{24cdx}{6cx}$.

Ex. 6. $5 \times 4 \times b = 20b$ First Numerator.
$3a \times 6 \times b = 18ab$ Second Numerator.
$(2b^2+3a) \times 6 \times 4 = 48b^2 + 72a$ Third Numerator.
And $6 \times 4 \times b = 24b$ Common Denominator.

Wherefore $\frac{20b}{24b}$, $\frac{18ab}{24b}$, and $\frac{48b^2+72a}{24b}$ are the fractions required, admitting however of reduction to *lower terms.*

Ex. 7. Ans. $\frac{4a+4b}{12a+12b}$, $\frac{6a^3+6a^2b}{12a+12b}$, and $\frac{24a^2+12b^2}{12a+12b}$; or, in *lowest* terms, $\frac{2a+2b}{6a+6b}$, $\frac{3a^3+3a^2b}{6a+6b}$, and $\frac{12a^2+6b^2}{6a+6b}$.

Ex. 8. Ans. $\frac{9b}{12a^2}$, $\frac{8ac}{12a^2}$, and $\frac{6ad}{12a^2}$.

(Page 182.)

To find the Greatest Common Measure of Fractions.

Ex. 3. Dividing the given divisor by b, it becomes $a+2$.

And $a+2)a^2 \quad * \quad -4(a-2$ Quotient.

$$\begin{array}{l} a^2+2a \\ \hline -2a-4 \\ -2a-4 \\ \hline \end{array}$$

$\cdot \quad \cdot$ *without remainder.*

$a+2$, therefore, is the greatest common measure.

(Key to Vol. I. page 183.)

Ex. 4. $a^4-b^4)a^5 \quad * \quad -a^3b^2(a$

a^5-ab^4

Divide by $ab^2 \text{- - - -})\underline{ab^4-a^3b^2}$ *remainder.*

Quot. $b^2-a^2=-a^2+b^2)a^4 \quad * \quad -b^4(-a^2-b^2$

$a^4-a^2b^2$

$a^2b^2-b^4$

$a^2b^2-b^4$

$\cdot \quad \cdot$

Here, $-a+b$ is the last divisor and greatest common measure; but it is evident that $a-b$ will likewise divide both terms of the given fraction without a remainder.

Ex. 5. The greatest common measure is 1; that is, the fraction is incommensurate. But, if instead of $\frac{a^3x+2a^2x^2+2ax^3+x^4}{5a^5+10a^4x+5a^2x^2}$ it was intended to submit $\frac{a^3x+2a^2x^2+2ax^3+x^4}{5a^5+10a^4x+5a^3x^2}$, the greatest common measure is $a+x$.

(Page 184.)

To reduce a Fraction to its Lowest Terms.

Ex. 3. The greatest common measure, by CASE IV. is $c-b$.

Hence $c-b)\,\frac{c^3-b^3}{c^4-b^2c^2}\,(\frac{c^2+bc+b^2}{c^3+bc^2}$ *lowest terms.*

Ex. 4. The greatest common measure being a^2-b^2,

It is $a^2-b^2)\frac{a^2-b^2}{a^4-b^4}(\frac{1}{a^2+b^2}$. *lowest terms.*

Ex. 5. The greatest common measure is $a-b$. Hence

$a-b)\,\frac{a^4-b^4}{a^3-3a^2b+3ab^2-b^3}\,(\frac{a^3+a^2b+ab^2+b^3}{a^2-2ab+b^2}$. *lowest terms.*

Ex. 6. The greatest common measure being $a^2+2ac+c^2$, It is

$a^2+2ac+c^2)\,\frac{3a^5+6a^4c+3a^3c^2}{a^3c+3a^2c^2+3ac^3+c^4}\,(\frac{3a^3}{ac+c^2}$. *lowest terms.*

(Key to Vol. I. page 184.)

Ex. 7. The greatest common measure is $a+b$.

Wherefore $a+b)\dfrac{a^3-ab^2}{a^2+2ab+b^2}(\dfrac{a^2-ab}{a+b}$. *lowest terms.*

ADDITION OF ALGEBRAIC FRACTIONS.

(Page 185.)

Ex. 4. $\dfrac{(4x\times5b)+(2x\times3a)}{3a\times5b}=\dfrac{20bx+6ax}{15ab}$. Ans.

Ex. 5. $\dfrac{(a\times4\times5)+(a\times3\times5)+(a\times3\times4)}{3\times4\times5}=\dfrac{47}{60}a$. Ans.

Ex. 6. $\dfrac{(2a-3)\times8+(5a\times4)}{4\times8}=\dfrac{9a-6}{8}$. Ans.

Ex. 7. $\dfrac{(11a+3)\times4+(18a-5)\times5}{5\times4}=6\dfrac{7}{10}a-\dfrac{13}{20}$, or

$6a+\dfrac{14a-13}{20}$. Ans.

Ex. 8. $\dfrac{(3a^2\times3b)+4b(a+b)+6a}{4b\times3b}=6a+\dfrac{9a^2+4a+4b}{12b}$. Ans.

Ex. 9.

$\dfrac{(5a\times5\times7)+(6a\times4\times7)+(3a+2)\times4\times5}{4\times5\times7}=2a+\dfrac{123a+40}{140}$. Ans.

Ex. 10. Ans. $2\dfrac{13}{24}a+3$.

Ex. 11. Ans. $10\dfrac{1}{8}a$.

SUBTRACTION OF ALGEBRAIC FRACTIONS.

(Page 186.)

Ex. 3. $\dfrac{10a}{9}-\dfrac{4a}{7}=\dfrac{70a-36a}{63}=\dfrac{34}{63}a$. Ans.

(Key to Vol. I. page 186.)

Ex. 4. $\frac{6a}{1}-\frac{3a}{4}=\frac{24a-3a}{4}=\frac{21}{4}a=5\frac{1}{4}a$. Ans.

Ex. 6. $\frac{5a}{4}-\frac{2a}{3}=\frac{15a-8a}{12}=\frac{7}{12}a$. Ans.

Ex. 7. $\frac{2a+6}{9}=\frac{10a+30}{45}$, and $\frac{4a+8}{5}=\frac{36a+72}{45}$.

But $\frac{36a+72-10a-30}{45}=\frac{26a+42}{45}$. Ans.

Ex. 8. The minuend reduced is $\left\{\frac{4ac+2a}{c}\right.$

The subtrahend reduced is $\left\{\frac{2ac-a+3b}{c}\right.$

Hence, *The difference required is* $\left\{\frac{2ac+3a-3b}{c}\right.$.

MULTIPLICATION OF ALGEBRAIC FRACTIONS.

(Page 187.)

Ex. 4. $\left.\begin{array}{ll}4a\times 6a=24a & \textit{Numerator.}\\ 3\times 5c=15c & \textit{Denominator.}\end{array}\right\}=\frac{8a}{5c}$ Ans.

Ex. 5. $\frac{3a}{4}\times\frac{4b^2}{3a}=b^2$. Ans.

Ex. 6. $\frac{3a}{b}\times\frac{8ac}{b}\times\frac{4ab}{3c}=\frac{32a^3}{b}$. Ans.

Ex. 7. Ans. $\frac{12a^3c+3a^3b}{2bc}$.

Ex. 8. Ans. $\frac{8a^4-4a^2b^2-4b^4}{3abc+3b^2c}$.

Ex. 9. Ans. $\frac{12a^2-3}{2a+b}$.

(Key to Vol. I. page 187.)

Ex. 10. Reducing the fractions to a common denominator, and collecting like terms, it will be $\frac{4a^3+2ax-x^2}{4a^2}\times\frac{4x^3-2ax+a^2}{4x^3}=\frac{4a^5-8a^4x+16a^3x^3-5a^2x^2+2a^3x+2ax^3+8ax^4-4x^5}{16a^2x^2}$, *the product required.*

DIVISION OF ALGEBRAIC FRACTIONS.

(Page 188.)

Ex. 5. $\frac{11}{12}\Big)\frac{3x}{4}\Big(\frac{36x}{44}=\frac{9}{11}x.$ Quotient.

Ex. 6. $\frac{3x}{1}\Big)\frac{6x^2}{5}\Big(\frac{6x^2}{15x}=\frac{2}{5}x.$ Quotient.

Ex. 7. $\frac{4x}{3}\Big)\frac{3x+1}{9}\Big(\frac{9x+3}{36}=\frac{1}{12}x+\frac{1}{4}.$ Quotient.

Ex. 8. $\frac{x}{3}\Big)\frac{4x}{2x-1}\Big(\frac{12x}{2x^2-x}=\frac{12}{2x-1}.$ Quotient.

Ex. 9. $\frac{3a}{5b}\Big)\frac{4x}{5}\Big(\frac{4bx}{3a}=\frac{1\frac{1}{3}bx}{a}.$ Quotient.

Ex. 10. $\frac{5ac}{6d}\Big)\frac{2a-b}{4cd}\Big(\frac{12a-6b}{20ac^2}=\frac{6a-3b}{10ac^2}.$ Quotient.

Ex. 11. Ans. $\frac{10a^5-10ab^4-10a^4b+10b^5}{6a^4-7a^3b-4a^2b^2+5ab^3}.$

INVOLUTION.

(Page 190.)

Ex. 1. $3a^2\times3a^2\times3a^2=27a^6$. The Cube required.

Ex. 2. $2a^2b\times2a^2b\times2a^2b\times2a^2b=16a^8b^4$. The Power required.

(Key to Vol. I. page 190.)

Ex. 3. Ans. $-64a^6b^9$. Ex. 4. Ans. $\frac{a^8x^4}{16b^8}$.

Ex. 5. Ans. $a^5-10a^4x+40a^3x^2-80a^2x^3+80ax^4-32x^5$.

Ex. 6.

Root	$2\sqrt{a}$		$2a^{\frac{1}{2}}$	*Root.*
Mul. by	$2\sqrt{a}$		$2a^{\frac{1}{2}}$	
Square	$4a$		$4a^{\frac{1}{1}}$	*Square.*
Mul. by	$2\sqrt{a}$		$2a^{\frac{1}{2}}$	
Cube	$8a\sqrt{a}$		$8a^{\frac{3}{2}}$	*Cube.*
Mul. by	$2\sqrt{a}$	or,	$2a^{\frac{1}{2}}$	
4th Power	$16aa=16a^2$		$16a^{\frac{2}{1}}$	*Biquadrate.*
Mul. by	$2\sqrt{a}$		$2a^{\frac{1}{2}}$	
5th Power	$32a^2\sqrt{a}$		$32a^{\frac{5}{2}}$	*Sursolid.*
Mul. by	$2\sqrt{a}$		$2a^{\frac{1}{2}}$	
6th Power	$64a^2a=64a^3$. Ans.		Ans. $64a^{\frac{3}{1}}$	*Sixth Power.*

EVOLUTION.

(Page 193.)

To determine the Roots of Simple Quantities.

Ex. 5. $\sqrt{2a^2b^4}=\sqrt{2}\times\sqrt{a^2}\times\sqrt{b^4}=ab^2\sqrt{2}$. Root required.

Ex. 6. $\sqrt[3]{-64a^3b^6}=\sqrt[3]{-64}\times\sqrt[3]{a^3}\times\sqrt[3]{b^6}=-4ab^2$. Root reqd.

Ex. 7. $\sqrt{\frac{8a^2b^2}{3c^3}}=\frac{\sqrt{4}\sqrt{2}\sqrt{a^2}\sqrt{b^2}}{\sqrt{c^2}\sqrt{3c}}=\frac{2ab}{c}\sqrt{\frac{2}{3c}}$. Root required.

Ex. 8. $\sqrt[4]{81a^4b^6}=\sqrt[4]{81}\sqrt[4]{a^4}\times\sqrt[4]{b^4}\times\sqrt[4]{b^2}=3ab\sqrt{b}$. Root reqd.

Ex. 9. $\sqrt[5]{-32a^5b^6}=\sqrt[5]{-32}\sqrt[5]{a^5}\times\sqrt[5]{b^5}\times\sqrt[5]{b}=-2ab\sqrt[5]{b}$. Ans.

(Key to Vol. I. page 195.)

To extract the Square Root of a Compound Quantity.

Ex. 3. $a^4+4a^3+6a^2+4a+1(a^2+2a+1$ the Root.
a^4

1st Divisor $2a^2+2a$) $4a^3+6a^2$ *Dividend.*
$4a^3+4a^2$

2d Divisor $2a^2+4a+1$) $2a^2+4a+1$ *Dividend.*
$2a^2+4a+1$

. . .

Ex. 4. $a^4-2a^3+2a^2-a+\frac{1}{4}(a^2-a+\frac{1}{2}$ the Root.
a^4

1st Divisor $2a^2-a$) $-2a^3+2a^2$ *Dividend.*
$-2a^3+a^2$

2d Divisor $2a^2-2a+\frac{1}{2}$) $a^2-a+\frac{1}{4}$ *Dividend.*
$a^2-a+\frac{1}{4}$

. . .

Ex. 5. $a^2-ab(a-\frac{b}{2}-\frac{b^2}{8a}-\frac{b^3}{16a^2}-$&c. Root reqd.
a^2

1st Divisor $2a-\frac{b}{2}$) $-ab$ *Dividend.*

$-ab+\frac{b^2}{4}$

2d Divisor $2a-b-\frac{b^2}{8a}$) $-\frac{b^2}{4}$ *Dividend.*
$-\frac{b^2}{4}+\frac{b^3}{8a}+\frac{b^4}{64a^2}$

3d Divisor $2a-b-\frac{b^2}{4a}-\frac{b^3}{16a^2}$) $-\frac{b^3}{8a}-\frac{b^4}{64a^2}$ *Dividend, &c.*

(Key to Vol. I. page 195.)

To determine the Roots of Powers in General.

Ex. 3. $a^2-2ab+2ax+b^2-2bx+x^2\ (a-b+x.$ Ans.
a^2

Divisor $2a)\cdot\ -2ab$ *Dividend.*

$a^2-2ab\quad +b^2$

Divisor $2a)\cdot\quad\cdot\ +2ax\ \cdot$ *Dividend.*

$a^2-2ab+2ax+b^2-2bx+x^2$

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Ex. 4. $a^6-3a^5+9a^4-13a^3+18a^2-12a+8(a^2-a+2.$ Ans.
a^6

Divis. $3a^4)\ \cdot-3a^5$ - - - - - - *Dividend.*

$a^6-3a^5+3a^4-\ a^3$

Divis. $3a^4)\ \cdot\quad\cdot\ +6a^4-12a^3$ - - *Remainder.*

$a^6-3a^5+9a^4-13a^3+18a^2-12a+8$

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Ex. 5. $81a^4-216a^3b+216a^2b^2-96ab^3+16b^4(3a-2b.$ Ans.
$81a^4$

Div. $108a^3)\ \cdot\ -216a^3b$ *Dividend.*

$81a^4-216a^3b+216a^2b^2-96ab^3+16b^4$

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Ex. 6. $a^5-10a^4+40a^3-80a^2+80a-32(a-2$ Root sought.
a^5

Divis. $5a^4)\ -10a^4$ *Dividend.*

$a^5-10a^4+40a^3-80a^2+80a-32$

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(Key to Vol. I. page 195.)

Ex. 7. $1-x^2\ (1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-$ &c. Root required.

1

Divisor $2)\ \cdot -x^2$ *Dividend.*

$1-x^2+\frac{x^4}{4}$ *Subtrahend.*

Divisor $2)\ \cdot\ \cdot -\frac{x^4}{4}$ *Rem. and Dividend.*

$1-x^2\ *\ +\frac{x^6}{8}+\frac{x^8}{64}$ *Subtrahend.*

Divisor $2)\ \cdot\ \cdot\ \cdot -\frac{x^6}{8}-\frac{x^8}{64}$ *Remainder.*

$1-x^2$ &c.+&c. *Subtrahend.*

Ex. 8. $1-x^3\ (1-\frac{x^3}{3}-\frac{x^6}{9}-$ &c. AD INFINITUM. Cube root reqd.

1

Div. $3)\ \cdot -x^3$ *Dividend.*

$1-x^3+\frac{x^6}{3}+\frac{x^9}{27}$ *Subtrahend.*

Div. $3)\ \cdot\ \cdot -\frac{x^6}{3}-\frac{x^9}{27}$ *Remainder.*

$1-x^3$ &c.+&c. *Subtrahend.*

SURDS.

(Page 196.)

To reduce a Rational Quantity to the Form of a Surd.

Ex. 3. $6\times6\times6=216$, wherefore $\sqrt[3]{216}$, or, $(216)^{\frac{1}{3}}$. form reqd.

(Key to Vol. I. page 196.)

Ex. 4. $\frac{1}{3}ab \times \frac{1}{3}ab = \frac{1}{9}a^2b^2$, wherefore $\sqrt{\frac{1}{9}a^2b^2}$, or, $(\frac{1}{9}a^2b^2)^{\frac{1}{2}}$. Ans.

Ex. 5. $2 \times 2 \times 2 \times 2 = 16$, wherefore $\sqrt[4]{16}$, or, $(16)^{\frac{1}{4}}$. Ans.

Ex. 6. $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{5}{3}} = \sqrt[3]{a^5}$. form required.

Ex. 7.
$(a+x) \times (a+x) = a^2 + 2ax + x^2$, hence $\sqrt{a^2+2ax+x^2}$. Ans.

Ex. 8. $\left\{(a-x) \times (a-x) \times (a-x)\right\}^{\frac{1}{3}} = \sqrt[3]{a^3 - 3a^2x + 3ax^2 - x^3}$.the form required.

(Page 197.)

To reduce Quantities to a Common Index.

Ex. 3. $\frac{1}{4})\frac{1}{3}(\frac{4}{3}$ the form for 4.
And $\frac{1}{4})\frac{1}{2}(\frac{4}{2} = \frac{2}{1}$ the form for 5.
But $4^4 = 256$, and $5^2 = 25$,

Hence $(256^{\frac{1}{3}})^{\frac{1}{4}}$, and $25^{\frac{1}{4}}$. Ans.

Ex. 4. $\frac{1}{6})\frac{1}{3}(\frac{6}{3} = \frac{2}{1}$ the form for a.
And $\frac{1}{6})\frac{1}{4}(\frac{6}{4} = \frac{3}{2}$ the form for x.

Hence $(a^2)^{\frac{1}{6}}$, and $(x^{\frac{3}{2}})^{\frac{1}{6}}$. Ans.

Ex. 5. For the form of the Square Root,

$$\left.\begin{array}{l} \frac{1}{2})\frac{2}{1}(\frac{4}{1} \text{ for the index of } a, \\ \text{And } \frac{1}{2})\frac{3}{1}(\frac{6}{1} \text{ for the index of } x, \end{array}\right\} \text{giving } \sqrt{a^4}, \text{ and } \sqrt{x^6}. \text{ Ans.}$$

For the form of the Cube Root,

$$\left.\begin{array}{l} \frac{1}{3})\frac{2}{1}(\frac{6}{1} \text{ for the index of } a, \\ \text{And } \frac{1}{3})\frac{3}{1}(\frac{9}{1} \text{ for the expon. of } x, \end{array}\right\} \text{hence } \sqrt[3]{a^6}, \text{ and } \sqrt[3]{x^9}. \text{ Ans.}$$

(Key to Vol. I. page 197.)

For the form of the nth Root,

$$\left.\begin{array}{r}\frac{1}{n})\frac{2}{1}(\frac{2n}{1}\text{ for the Index of }a,\\ \text{And }\frac{1}{n})\frac{3}{1}(\frac{3n}{1}\text{ for the Index of }x,\end{array}\right\}\text{ wherefore }\sqrt[n]{a^{2n}},\text{ and }\sqrt[n]{x^{3n}}.$$

But it is evident that $a^{\frac{4}{2}}=a^2$, and that $x^{\frac{6}{2}}=x^3$, consequently the given quantities have the same radical sign, namely 1, without any operation.

Ex. 6. $\frac{1\times2}{3\times2}=\frac{2}{6}$ for the form of $(a+x)$, And

$\frac{1\times3}{3\times2}=\frac{3}{6}$ for the form of $(a-x)$

$$\left.\begin{array}{r}\text{Consequently }(a^2+2ax+x^2)^{\frac{1}{6}}\\ \text{And }(a^3-3a^2x+3ax^2-x^3)^{\frac{1}{6}}\end{array}\right\}\text{ Ans.}$$

Ex. 7. $\frac{1\times4}{2\times4}=\frac{4}{8}$, and $\frac{1\times2}{2\times4}=\frac{2}{8}$

$$\left.\begin{array}{r}\text{Therefore }(a^4+4a^3b+6a^2b^2+4ab^3+b^4)^{\frac{1}{8}}\\ \text{And }(a^2-2ab+b^2)^{\frac{1}{8}}\end{array}\right\}\text{ Ans.}$$

(Page 198.)

To reduce Complicated Surds to Simpler Terms.*

Ex. 3. $\sqrt{75}=\sqrt{25}\times\sqrt{3}=5\sqrt{3}$. Ans.

Ex. 4. $\sqrt{\frac{44}{75}}=\frac{\sqrt{4}\sqrt{11}}{\sqrt{25}\sqrt{3}}=\frac{2}{5}\times\frac{\sqrt{11}}{\sqrt{3}}=\frac{2}{5}\times\frac{\sqrt{33}}{\sqrt{9}}=\frac{2}{15}\sqrt{33}$. Ans.

Ex. 5. $\sqrt[3]{189}=\sqrt[3]{27}\sqrt[3]{7}=3\sqrt[3]{7}$. Ans.

Ex. 6. $\sqrt[3]{\frac{135}{32}}=\frac{\sqrt[3]{27}\sqrt[3]{5}}{\sqrt[3]{8}\ \sqrt[3]{4}}=\frac{3\sqrt[3]{5}}{2\sqrt[3]{4}}=\frac{3}{2}\times\frac{\sqrt[3]{10}}{\sqrt[3]{8}}=\frac{3}{4}\sqrt[3]{10}$. Ans.

* Quantities consisting of a *rational* and an *irrational* part involved together are termed Complicated Surds.

(Key to Vol. I. page 198.)

Ex. 7. $\sqrt{75a^2b} = \sqrt{25}\sqrt{3}\sqrt{a^2}\sqrt{b} = 5a\sqrt{3b}$. Ans.

ADDITION OF SURDS.

(Page 199.)

Ex. 3.
$$\begin{aligned} & \sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3} \\ \text{And } & \sqrt{48} = \sqrt{16}\sqrt{3} = 4\sqrt{3} \\ & \overline{7\sqrt{3}} \text{ Sum required.} \end{aligned}$$

Ex. 4.
$$\begin{aligned} & \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2} \\ \text{And } & \sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2} \\ & \overline{11\sqrt{2}} \text{ Sum required.} \end{aligned}$$

Ex. 5. $\sqrt{\frac{3}{5}} = \sqrt{\frac{45}{75}} = \frac{\sqrt{9}\sqrt{5}}{\sqrt{25}\sqrt{3}} = \frac{3}{5} \times \frac{\sqrt{5}}{\sqrt{3}} = \frac{3}{5}\sqrt{\frac{15}{9}} = \frac{3}{15}\sqrt{15}$.

And $\sqrt{\frac{1}{15}} = \sqrt{\frac{5}{75}} = \frac{\sqrt{1}\sqrt{5}}{\sqrt{25}\sqrt{3}} = \frac{1}{5} \times \frac{\sqrt{5}}{\sqrt{3}} = \frac{1}{5}\sqrt{\frac{15}{9}} = \frac{1}{15}\sqrt{15}$.

Sum and Answer $\frac{4}{15}\sqrt{15}$.

But $\frac{4}{15}\sqrt{15} = \frac{4}{1} \times \frac{\sqrt{15}}{\sqrt{225}} = 4\sqrt{\frac{1}{15}}$ the *other Ans.* given in the COURSE.

Ex. 6.
$$\begin{aligned} & \sqrt[3]{56} = \sqrt[3]{8}\sqrt[3]{7} = 2\sqrt[3]{7} \\ \text{And } & \sqrt[3]{189} = \sqrt[3]{27}\sqrt[3]{7} = 3\sqrt[3]{7} \\ & \overline{5\sqrt[3]{7}} \text{ Sum required.} \end{aligned}$$

(Key to Vol. I. page 199.)

Ex. 7. $\sqrt[3]{\frac{1}{4}}=\frac{\sqrt[3]{128}}{\sqrt[3]{512}}=\frac{\sqrt[3]{8}\sqrt[3]{16}}{\sqrt[3]{64}\sqrt[3]{8}}=\frac{1}{4}\sqrt[3]{16}=\frac{1}{4}\sqrt[3]{8}\sqrt[3]{2}=\frac{1}{2}\sqrt[3]{2}.$

And $\sqrt[3]{\frac{1}{32}}=\frac{\sqrt[3]{16}}{\sqrt[3]{512}}=\frac{\sqrt[3]{1}\sqrt[3]{16}}{\sqrt[3]{64}\sqrt[3]{8}}=\frac{1}{8}\sqrt[3]{16}=\frac{1}{8}\sqrt[3]{8}\sqrt[3]{2}=\frac{1}{4}\sqrt[3]{2}.$

Sum and Answer $\frac{3}{4}\sqrt[3]{2}.$

Ex. 8. $3\sqrt{a^2b}=3\sqrt{a^2}\sqrt{b}=3a\sqrt{b}$

And $5\sqrt{16a^4b}=5\sqrt{16}\sqrt{a^4}\sqrt{b}=20a^2\sqrt{b}$

Sum $(20a^2+3a)\sqrt{b}.$ Ans.

SUBTRACTION OF SURDS.

Ex. 3. $\sqrt{75}=\sqrt{25}\sqrt{3}=5\sqrt{3}$

And $\sqrt{48}=\sqrt{16}\sqrt{3}=4\sqrt{3}$

Difference $\sqrt{3}.$ Ans.

Ex. 4. $\sqrt[3]{256}=\sqrt[3]{64}\sqrt[3]{4}=4\sqrt[3]{4}$

And $\sqrt[3]{32}=\sqrt[3]{8}\sqrt[3]{4}=2\sqrt[3]{4}$

Difference $2\sqrt[3]{4}.$ Ans.

Ex. 5. $\sqrt{\frac{3}{4}}=\sqrt{\frac{9}{12}}=\frac{\sqrt{9}\sqrt{1}}{\sqrt{4}\sqrt{3}}=\frac{3}{2}\sqrt{\frac{1}{3}}=\frac{3}{2}\sqrt{\frac{3}{9}}=\frac{1}{2}\sqrt{3}$

And $\sqrt{\frac{1}{3}}=\sqrt{\frac{4}{12}}=\frac{\sqrt{4}\sqrt{1}}{\sqrt{4}\sqrt{3}}=\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{9}}=\frac{1}{3}\sqrt{3}$

Difference $\frac{1}{6}\sqrt{3}.$ Ans.

(Key to Vol. I. page 199.)

Ex. 6. $$\sqrt{\frac{2}{3}}=\sqrt{\frac{16}{24}}=\frac{\sqrt{16}\sqrt{1}}{\sqrt{4}\sqrt{6}}=2\sqrt{\frac{1}{6}}=2\sqrt{\frac{6}{36}}=\frac{4}{12}\sqrt{6}$$

And $$\sqrt{\frac{3}{8}}=\sqrt{\frac{9}{24}}=\frac{\sqrt{9}\sqrt{1}}{\sqrt{4}\sqrt{6}}=\frac{3}{2}\sqrt{\frac{1}{6}}=\frac{3}{2}\sqrt{\frac{6}{36}}=\frac{3}{12}\sqrt{6}$$

Difference $\frac{1}{12}\sqrt{6}$. Ans.

Ex. 7. $$\sqrt[3]{2\tfrac{7}{9}}=\sqrt[3]{\frac{25}{9}}=\frac{\sqrt[3]{125}}{\sqrt[3]{45}}=\frac{5}{\sqrt[3]{45}}$$

And $$\sqrt[3]{\frac{3}{5}}=\frac{\sqrt[3]{27}}{\sqrt[3]{45}}=\frac{3}{\sqrt[3]{45}}$$

Difference $\frac{2}{\sqrt[3]{45}}$

But $$\frac{2}{\sqrt[3]{45}}=\frac{\sqrt[3]{8}}{\sqrt[3]{45}}=\frac{\sqrt[3]{8}\sqrt[3]{2025}}{45}=\frac{2}{45}\sqrt[3]{27}\sqrt[3]{75}=\frac{2}{15}\sqrt[3]{75}.$$ Ans.

Ex. 8. $\sqrt{24a^2b^2}=\sqrt{4}\sqrt{6}\sqrt{a^2}\sqrt{b^2}=2ab\sqrt{6}$

And $\sqrt{54b^4}=\sqrt{9}\sqrt{6}\sqrt{b^4}=$ - - - $3b^2\sqrt{6}$

Therefore $\begin{cases} {}^{*}\text{when } 2ab > 3b^2 \text{ - - } (2ab-3b^2)\sqrt{6} \\ \text{when } 2ab < 3b^2 \text{ - - } (3b^2-2ab)\sqrt{6} \end{cases}$ Ans.

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* It may be necessary to mention that > signifies *greater than*, and < *less than*.

MULTIPLICATION OF SURDS.

(Page 200.)

Ex. 3. $3\times2=6$ the Product of the rational Quantities.

And $\sqrt{2}\times\sqrt{8}=\sqrt{16}=4$ the Product of the Surd Quantities.

Consequently 24 *Product* required.

(Key to Vol. I. page 200.)

Ex. 4. $\frac{1}{3}\times\frac{3}{4}=\frac{1}{4}$ the Product of the rational Quantities.

And $\sqrt[3]{4}\times\sqrt[3]{12}=\sqrt[3]{48}=\sqrt[3]{8}\sqrt[3]{6}=2\sqrt[3]{6}$ Product of the Surds.

Wherefore $\frac{1}{4}\times2\sqrt[3]{6}=\frac{1}{2}\sqrt[3]{6}$. Ans.

Ex. 5. $\frac{5}{3}\times\frac{9}{10}=\frac{3}{2}$ the Product of the rational Quantities.

And $\sqrt{\frac{3}{8}}\times\sqrt{\frac{2}{5}}=\sqrt{\frac{6}{40}}=\frac{\sqrt{1}\sqrt{6}}{\sqrt{4}\sqrt{10}}=\frac{1}{2}\sqrt{\frac{6}{10}}=\frac{1}{2}\frac{\sqrt{60}}{10}=\frac{1}{20}\sqrt{60}$

But $\frac{1}{20}\sqrt{60}=\frac{1}{10}\sqrt{15}$ the Product of the Surd Quantities.

Therefore $\frac{3}{2}\times\frac{1}{10}\sqrt{15}=\frac{3}{20}\sqrt{15}$. Ans.

Ex. 6. $2\times3=6$ the Product of the rational Parts.

And $\sqrt[3]{14}\times\sqrt[3]{4}=\sqrt[3]{56}=\sqrt[3]{8}\sqrt[3]{7}=2\sqrt[3]{7}$ Prod. of the Surds.

Wherefore $12\sqrt[3]{7}$. Ans.

Ex. 7. $2\times1=2$ the Product of the Co-efficients.

And $a^{\frac{2}{3}}\times a^{\frac{4}{3}}=a^{\frac{6}{3}}=a^2$ the Product of the Surds.

Consequently $2a^2$. Ans.

Ex. 8. $\frac{1}{3}=\frac{4}{12}$, and $\frac{3}{4}=\frac{9}{12}$, the new indices with a com. radical.

But $\sqrt[12]{\overline{a+b}^4}\times\sqrt[12]{\overline{a+b}^9}=(a+b)^{\frac{13}{12}}$. That is,

$(a^{13}+13a^{12}b+78a^{11}b^2+286a^{10}b^3+715a^9b^4+\&c.\ \&c.)^{\frac{1}{12}}$. Ans.

Ex. 9.

$$\begin{array}{llll}
2x & +\sqrt{b} & & \textit{Multiplicand.} \\
2x & -\sqrt{b} & & \textit{Multiplier.} \\
\hline
4x^2 & +2x\sqrt{b} & & \text{Prod. by } 2x. \\
 & -2x\sqrt{b} & -b & \text{Prod. by } -\sqrt{b}. \\
\hline
4x^2 & * & -b & \text{PROD. by } (2x-\sqrt{b}) \text{ which was reqd.} \\
\hline
\end{array}$$

(Key to Vol. I. page 200.)

Ex. 10. $(a+2\sqrt{b})^{\frac{1}{2}}$ *Multiplicand.*
$(a-2\sqrt{b})^{\frac{1}{2}}$ *Multiplier.*

$\sqrt{(a^2 \quad * \quad -4b)}$ Product required.

Ex. 11. $2x^{\frac{1}{n}} \times 3x^{\frac{1}{n}} = 6(x^2)^{\frac{1}{n}}$ or $6\sqrt[n]{x^2}$. Ans.

Or if, as in the Sixth Edition,* the question be $2x^{\frac{1}{n}}$ and $3x^{\frac{1}{m}}$, *First* reducing the Surds to a common radical, it is

$2x^{\frac{m}{mn}} \times 3x^{\frac{n}{mn}}$.

Next, $2x^{m} \times 3x^{n} = 6x^{m+n}$, therefore, subscribing the common index, $6x^{\frac{m+n}{mn}} = 6\sqrt[mn]{x^{m+n}}$. Ans.

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* There happens to be two Sixth Editions of Hutton's Course (if both can be well termed Editions,) the one printed in 1811, and the other (whereof a part, at least, was certainly printed subsequently to the corresponding part of the former) purporting to have been printed in 1815. Here we allude to the latter Edition, whereas in page 30 of the Key the allusion applies to the former.

Ex. 12. $4x^{\frac{1}{n}} \times 2y^{\frac{1}{n}} = 8(xy)^{\frac{1}{n}}$, or $8\sqrt[n]{xy}$. Ans.

DIVISION OF SURDS.

(Page 201.)

Ex. 3. $4 \div 2 = 2$ the Quotient of the Co-efficients.
And $\sqrt{50} \div \sqrt{5} = \sqrt{10}$. Therefore $2\sqrt{10}$. Ans.

Ex. 4. $6 \div 3 = 2$ the Quotient of the Co-efficients.
And $\sqrt[3]{100} \div \sqrt[3]{5} = \sqrt[3]{20}$. Therefore $2\sqrt[3]{20}$. Ans.

Ex. 5. $\frac{5}{6} \div \frac{3}{4} = \frac{10}{9}$ the Quotient of the Co-efficients.

And $\sqrt{\frac{1}{50}} \div \sqrt{\frac{2}{5}} = \sqrt{\frac{5}{100}} = \frac{1}{10}\sqrt{5}$. Quotient of the Surds.

But $\frac{10}{9} \times \frac{1}{10}\sqrt{5} = \frac{1}{9}\sqrt{5}$. Ans.

(Key to Vol. I. page 201.)

Ex. 6. $\frac{3}{4} \div \frac{3}{5} = \frac{5}{4}$ the Quotient of the Co-efficients.

And $\sqrt[3]{\frac{3}{16}} \div \sqrt[3]{\frac{2}{5}} = \sqrt[3]{\frac{15}{32}} = \sqrt[3]{\frac{30}{64}} = \frac{1}{4}\sqrt[3]{30}$. Quot. of the Surds.

But $\frac{5}{4} \times \frac{1}{4}\sqrt[3]{30} = \frac{5}{16}\sqrt[3]{30}$. Ans.

Ex. 7. $\frac{4}{5} \div \frac{2}{3} = \frac{6}{5}$ the Quotient of the Co-efficients.

And $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ for the exponent of the Quotient Surd.

Therefore $\frac{6}{5}a^{\frac{1}{6}}$. Ans.

Ex. 8. $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$ for the exponent of a in the Quotient.

Therefore $a^{\frac{2}{3}}$, or $\sqrt[3]{a^2}$. Ans.

Ex. 9. $3 \div 4 = \frac{3}{4}$ the Quotient of the Co-efficients.

And $\frac{1}{n}$, $\frac{1}{m}$, reduced to a common radical are $\frac{m}{mn}$, and $\frac{n}{mn}$.

Therefore $\frac{3}{4}a^{\frac{m-n}{mn}}$ Ans.

INVOLUTION OF SURDS.

Ex. 4. $2\sqrt[3]{2} \times 2\sqrt[3]{2} = 4\sqrt[3]{4}$. Ans.

Ex. 5. $\sqrt{3} \times \sqrt{3} \times \sqrt{3} = \sqrt{3^3} = \sqrt{27} = \sqrt{9}\sqrt{3} = 3\sqrt{3}$. Ans.

Ex. 6. $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$ the power required of the Co-efficient.

And, $\sqrt{3} \times \sqrt{3} \times \sqrt{3} = \sqrt{27} = 3\sqrt{3}$ the Cube of the Surd.

But $\frac{1}{27} \times 3\sqrt{3} = \frac{1}{9}\sqrt{3}$. Ans.

(Key to Vol. I. page 201.)

Ex. 7. $2^{\frac{1}{2}}\times2^{\frac{1}{2}}\times2^{\frac{1}{2}}\times2^{\frac{1}{2}}=2^2$ or $16^{\frac{1}{2}}=4$ the power of the Surd.

And $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{16}$ the power of the Co-efficient.

But $\frac{1}{16}\times4=\frac{1}{4}$. Ans.

(Page 202.)

Ex. 8. Ans. $\sqrt[n]{a^m}$, or $a^{\frac{m}{n}}$.

Ex. 9. $2+\sqrt{3}$ - - - - - Given Root.
$2+\sqrt{3}$ - - - - - Multiplier.

$4+2\sqrt{3}$ - - - - *Prod. by* 2.
$+2\sqrt{3}+3$ - - *Prod. by* $\sqrt{3}$.

Sum $4+4\sqrt{3}+3=7+4\sqrt{3}$, the Square required.

EVOLUTION OF SURDS.

Ex. 3. $6^3=216$.

Now $\sqrt{216}=\sqrt{36}\sqrt{6}=6\sqrt{6}$. Ans.

Ex. 4. $\sqrt[3]{\frac{1}{8}a^3b}=\sqrt[3]{\frac{1}{8}}\sqrt[3]{a^3}\sqrt[3]{b}=\frac{1}{2}a\sqrt[3]{b}$. Ans.

Ex. 5. $\sqrt[4]{16a^2}=\sqrt[4]{16}\sqrt[4]{a^2}=2\sqrt{a}$. Ans.

Ex. 6. $\frac{1}{n}\div\frac{m}{1}=\frac{1}{mn}$, therefore $x^{\frac{1}{mn}}$. Ans.

Ex. 7. $a^2-6a\sqrt{b}+9b\ (a-3\sqrt{b}$ Root required.
a^2

Divisor $2a)\cdot -6a\sqrt{b}$ *Dividend.*

$a^2-6a\sqrt{b}+9b$.

(Key to Vol. I. page 204.)

INFINITE SERIES.

Fractional Quantities reduced to Infinite Series by Division.

Ex. 3.

$$a+c)\,b\ \text{- - -}\text{(}\frac{b}{a}-\frac{bc}{a^2}+\frac{bc^2}{a^3}-\text{\&c.}=\frac{b}{a}\times\text{(}1-\frac{c}{a}+\frac{c^2}{a^2}-\text{\&c.)}\quad\text{Ans.}$$

$$\begin{array}{l} b+\frac{bc}{a}\\ \hline \cdot\ -\frac{bc}{a}\\ -\frac{bc}{a}-\frac{bc^2}{a^2}\\ \hline \quad\cdot\ +\frac{bc^2}{a^2}\\ \quad+\frac{bc^2}{a^2}+\frac{bc^3}{a^3}\\ \hline \qquad\cdot\ -\frac{bc^3}{a^3}\ \text{remainder.}\\ \hline \end{array}$$

Ex. 4. $a-b)\,a\ \text{- - - - -}\text{(}1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}+\frac{b^4}{a^4}+\text{\&c.}$ Ans.

$$\begin{array}{l} a-b\\ \hline \cdot\ +b\\ +b-\frac{b^2}{a}\\ \hline \quad\cdot\ +\frac{b^2}{a}\\ \quad+\frac{b^2}{a}-\frac{b^3}{a^2}\\ \hline \qquad\cdot\ +\frac{b^3}{a^2}\cdot\ \text{\&c.}\\ \hline \end{array}$$

(Key to Vol. I. page 204.)

Ex. 5. $1+x)1-x\ (1-2x+2x^2-2x^3+2x^4-\&c.$ series reqd.

$$\begin{array}{l}
\underline{1+x} \\
\cdot -2x \\
\underline{-2x-2x^2} \\
\cdot\quad +2x^2 \\
\underline{\quad +2x^2+2x^3} \\
\cdot\quad -2x^3 \\
\underline{\quad -2x^3-2x^4} \\
\cdot\quad \underline{+2x^4.\ \&c.}
\end{array}$$

Ex. 6. $a^2+2ab+b^2)\,a^2\text{ - - - - - - }(1-\frac{2b}{a}+\frac{3b^2}{a^2}-\frac{4b^3}{a^3}+\ \&c.$ Ans.

$$\begin{array}{l}
\underline{a^2+2ab+b^2} \\
\cdot -2ab-b^2 \\
\underline{-2ab-4b^2-\frac{2b^3}{a}} \\
\cdot\quad \underline{+3b^2+\frac{2b^3}{a}.\ \&c.}
\end{array}$$

Ex. 7. $1+1)1\text{ - - - - }(1-1+1-1+\ \&c.$ the series required.

$$\begin{array}{l}
\underline{1+1} \\
\cdot -1 \\
\underline{-1-1} \\
\cdot\quad +1 \\
\underline{\quad +1+1} \\
\cdot\quad -1 \\
\underline{\quad -1-1} \\
\cdot\quad \underline{+1}\ \text{remainder.}
\end{array}$$

(Key to Vol. I. page 205.)

Compound Surds reduced in Infinite Series.

Ex. 2. $1+1(1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{5}{128}+\&c.$ Ans.

$$1$$

$$\overline{}$$

$$2+\tfrac{1}{2})\cdot+1$$

$$+1+\tfrac{1}{4}$$

$$\overline{\phantom{+1+\tfrac{1}{4}}}$$

$$2+1-\tfrac{1}{8})\quad\cdot-\tfrac{1}{4}$$

$$-\tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{64}$$

$$\overline{\phantom{-\tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{64}}}$$

$$\cdot+\tfrac{1}{8}-\tfrac{1}{64}.\ \&c.$$

Ex. 3. $1-1(1-\frac{1}{2}-\frac{1}{8}-\frac{1}{16}-\frac{5}{128}-\&c.$ Root required.

$$1$$

$$\overline{}$$

$$2-\tfrac{1}{2})\cdot-1$$

$$-1+\tfrac{1}{4}$$

$$\overline{\phantom{-1+\tfrac{1}{4}}}$$

$$2-1-\tfrac{1}{8})\quad\cdot-\tfrac{1}{4}$$

$$-\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{64}$$

$$\overline{\phantom{-\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{64}}}$$

$$\cdot-\tfrac{1}{8}-\tfrac{1}{64}.\ \&c.$$

Ex. 4. $a^2+x\,(a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\&c.$ Root in the series.

$$a^2$$

$$\overline{}$$

$$2a+\frac{x}{2a})\cdot\ +x$$

$$+x+\frac{x^2}{4a^2}$$

$$\overline{\phantom{+x+\frac{x^2}{4a^2}}}$$

$$2a+\frac{x}{a}-\frac{x^2}{8a^3})\quad\cdot-\frac{x^2}{4a^2}$$

$$-\frac{x^2}{4a^2}-\frac{x^3}{8a^4}+\frac{x^4}{64a^6}$$

$$\overline{\phantom{-\frac{x^2}{4a^2}-\frac{x^3}{8a^4}+\frac{x^4}{64a^6}}}$$

$$\cdot\ +\frac{x^3}{8a^4}-\frac{x^4}{64a^6},\ \&c.$$

(Key to Vol. I. page 205.)

Ex. 5. $a^2-2bx-x^2\left(a-\frac{2bx+x^2}{2a}-\frac{4b^2x^2+4bx^3+x^4}{8a^3}-\&c.\right.$

$$\begin{array}{l|l} & a^2 \\ \hline 2a-\frac{2bx+x^2}{2a}\ ) & \cdot\ -2bx-x^2 \\ & -2bx-x^2+\frac{4b^2x^2+4bx^3+x^4}{4a^2} \\ \hline & \qquad -\frac{4b^2x^2+4bx^3+x^4}{4a^2}.\ \&c. \end{array}$$

Wherefore the series is $a-\frac{2bx+x^2}{2a}-\frac{4b^2x^2+4bx^3+x^4}{8a^3}$ &c. IN INFINITUM.

(Page 207.)

The Extraction of Roots of Binomial Quantities, and the Conversion of Binomial Surds to Infinite Series.

Ex. 3. Here $\frac{a^2}{a-x}=a^2(a-x)^{-1}$, and

$\text{P}=a,$

$\text{Q}=\frac{-x}{a},$

$\frac{m}{n}=\frac{-1}{1}$; therefore

$\text{P}^{\frac{m}{n}}=(a)^{-1}=a^{-1}=\frac{1}{a}=\text{A}$, the 1st term of a series.

$\frac{m}{n}\text{AQ}=\frac{-1}{1}\times\frac{1}{a}\times\frac{-x}{a}=\frac{x}{a^2}=\text{B}$, the 2d term.

$\frac{m-n}{2n}\text{BQ}=\frac{-2}{2}\times\frac{x}{a^2}\times\frac{-x}{a}=\frac{2x^2}{2a^3}=\frac{x^2}{a^3}=\text{C}$, 3d term.

$\frac{m-2n}{3n}\text{CQ}=$&c. &c. $\frac{x^3}{a^4}=\text{D}$, &c.

(Key to Vol. I. page 207.)

But $a^2\times(\frac{1}{a}+\frac{x}{a^2}+\frac{x^2}{a^3}+\frac{x^3}{a^4}+\frac{x^4}{a^5}+\frac{x^5}{a^6}+\&c.)=a+x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\frac{x^5}{a^4}+\&c.$ Ans.

Ex. 4. $(\frac{1}{a^2+x^2})^{\frac{1}{2}}=1\times(a^2+x^2)^{-\frac{1}{2}}=(a^2+x^2)^{-\frac{1}{2}}$.

Also $\text{P}=a^2$,

$\text{Q}=\frac{x^2}{a^2}$,

$\frac{m}{n}=\frac{-1}{2}$; therefore

$\text{P}^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=\frac{1}{a}=\text{A}$, the 1st term of the series sought.

$\frac{m}{n}\text{AQ}=\frac{-1}{2}\times\frac{1}{a}\times\frac{x^2}{a^2}=\frac{-x^2}{2a^3}=\text{B}$, the 2d term.

$\frac{m-n}{2n}\text{BQ}=\frac{-3}{4}\times\frac{-x^2}{2a^3}\times\frac{x^2}{a^2}=\frac{3x^4}{8a^5}=\text{C}$, the 3d term, &c.

Hence the series is $\frac{1}{a}-\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}-\frac{5x^6}{16a^7}+\&c.$ IN INFINITUM.

Ex. 5. Here $\frac{a^2}{(a-b)^2}=a^2\times(a-b)^{-2}$. and

$\text{P}=a$,

$\text{Q}=\frac{-b}{a}$,

$\frac{m}{n}=\frac{-2}{1}$; therefore

$\text{P}^{\frac{m}{n}}=(a)^{-2}=\frac{1}{a^2}=\text{A}$, the 1st term of a series to be $\times^{\text{ed}}$ by a^2.

$\frac{m}{n}\text{AQ}=\frac{-2}{1}\times\frac{1}{a^2}\times\frac{-b}{a}=\frac{2b}{a^3}=\text{B}$, the 2d term.

$\frac{m-n}{2n}\text{BQ}=\frac{-3}{2}\times\frac{2b}{a^3}\times\frac{-b}{a}=\frac{3b^2}{a^4}=\text{C}$, the 3d term, &c.

And $a^2\times(\frac{1}{a^2}+\frac{2b}{a^3}+\frac{3b^2}{a^4}+\&c.)=1+\frac{2b}{a}+\frac{3b^2}{a^2}+\frac{4b^3}{a^3}+\frac{5b^4}{a^4}+\&c.$ Ans

(Key to Vol. I. page 207.)

Ex. 6. Here, $P=a^2$,

$$Q=\frac{-x^2}{a^2}, \text{ and}$$

$$\frac{m}{n}=\frac{1}{2}; \text{ therefore}$$

$$P^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A.$$

$$\frac{m}{n}AQ=\tfrac{1}{2}\times\frac{a}{1}\times\frac{-x^2}{a^2}=\frac{-x^2}{2a}=B. \&c.$$

Hence $a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-$&c. the Root reqd.

Ex. 7. Here, $P=a^3$,

$$Q=\frac{-b^3}{a^3}, \text{ and}$$

$$\frac{m}{n}=\frac{1}{3}; \text{ therefore}$$

$P^{\frac{m}{n}}=(a^3)^{\frac{1}{3}}=a=A$, the 1st term of the series.

$\frac{m}{n}AQ=\frac{1}{3}\times\frac{a}{1}\times\frac{-b^3}{a^3}=\frac{-b^3}{3a^2}=B$, the 2d term, &c.

Hence $a-\frac{b^3}{3a^2}-\frac{b^6}{9a^5}-\frac{5b^9}{81a^8}-$&c. the Cube Root required.

Ex. 8. In this Example, $P=a^5$, $Q=\frac{x^5}{a^5}$, and $\frac{m}{n}=\frac{1}{5}$; therefore

$P^{\frac{m}{n}}=(a^5)^{\frac{1}{5}}=a=A$, the 1st term of the series.

$\frac{m}{n}AQ=\frac{1}{5}\times\frac{a}{1}\times\frac{x^5}{a^5}=\frac{x^5}{5a^4}=B$, the 2d term.

$\frac{m-n}{2n}BQ=\frac{-4}{10}\times\frac{x^5}{5a^4}\times\frac{x^5}{a^5}=\frac{-4x^{10}}{50a^9}=\frac{-2x^{10}}{25a^9}=C$, the 3d term.

Wherefore $a+\frac{x^5}{5a^4}-\frac{2x^{10}}{25a^9}+\frac{6x^{15}}{125a^{14}}$ &c. Ans.*

---

* *The sign of the 3d Term of the Answer given in the Course is different from the sign here given.*

(Key to Vol. I. page 208.)

Ex. 9. $\sqrt{\frac{a^2-b^2}{a^2+b^2}}=(a^2-b^2)^{\frac{1}{2}}\times(a^2+b^2)^{-\frac{1}{2}}$.

Now $(a^2-b^2)^{\frac{1}{2}}=a-\frac{b^2}{2a}-\frac{b^4}{8a^3}-$&c.

And $(a^2+b^2)^{-\frac{1}{2}}=\frac{1}{a}-\frac{b^2}{2a^3}+\frac{b^4}{8a^5}$ &c. Whence

$1-\frac{b^2}{a^2}+\frac{b^4}{2a^4}-\frac{b^6}{2a^6}$ &c. the Square Root required.

But, as this method is tedious and complex, it is better to extract the Square Root of the Quotient in an Infinite Series. Thus,

$$a^2+b^2)\,a^2-b^2(1-\frac{2b^2}{a^2}+\frac{2b^4}{a^4}-\frac{2b^6}{a^6}+\frac{2b^8}{a^8}-\frac{2b^{10}}{a^{10}}+\frac{2b^{12}}{a^{12}}-\text{\&c. Quot.}$$

And $\sqrt{(1-\frac{2b^2}{a^2}+\frac{2b^4}{a^4}-\text{\&c.})}=1-\frac{b^2}{a^2}+\frac{b^4}{2a^4}-\frac{b^6}{2a^6}+$&c. Ans.*

as before.

* This Answer, however, differs from that given by Dr. Hutton.

Ex. 10. $\sqrt[3]{\frac{a^3}{a^3+b^3}}=\sqrt[3]{a^3}\times(a^3+b^3)^{-\frac{1}{3}}=a(a^3+b^3)^{-\frac{1}{3}}$.

$$\left.\begin{array}{l}P=a^3\\ Q=\frac{b^3}{a^3}\\ \frac{m}{n}=\frac{-1}{3}\end{array}\right\}\begin{array}{l}P\frac{m}{n}=(a^3)^{-\frac{1}{3}}=\frac{1}{\sqrt[3]{a^3}}=\frac{1}{a}=\text{A, the 1st term of a series.}\\ \frac{m}{n}AQ=\frac{-1}{3}\times\frac{1}{a}\times\frac{b^3}{a^3}=\frac{-b^3}{3a^4}=\text{B, the 2d term.}\\ \frac{m-n}{2n}BQ=\frac{-4}{6}\times\frac{-b^3}{3a^4}\times\frac{b^3}{a^3}=\frac{2b^6}{9a^7}=\text{C, the 3d term.}\end{array}$$

$$\frac{m-2n}{3n}CQ=\frac{-7}{9}\times\frac{2b^6}{9a^7}\times\frac{b^3}{a^3}=\frac{-14b^9}{81a^{10}}=\text{D, the 4th term, \&c.}$$

But $a\times(\frac{1}{a}-\frac{b^3}{3a^4}+\frac{2b^6}{9a^7}-\text{\&c.})=1-\frac{b^3}{3a^3}+\frac{2b^6}{9a^6}-\frac{14b^9}{81a^9}+$&c. Ans.*

* By which it may be perceived that there is an Error in the Numerator of the 3d Term of Hutton's Answer.

(Key to Vol. I. page 209.)

ARITHMETICAL PROPORTION.

Ex. 3. The 1st term is 1, the last term 100, also the number of terms 100.

But $(1+100)\times 50=5050$ *Sum required.*

(Page 210.)

Ex. 4. $(98\times 2)+1=197$ the last term of the series.

And $\frac{1+197}{2}\times 99=9801$ *the Sum required.*

Ex. 5. $10-(\frac{1}{3}\times 20)=3\frac{1}{3}$ the 1st term of the series.

And $\frac{10+3\frac{1}{3}}{2}\times 21=140$ *the Sum required.*

Ex. 6. Here the 1st term is 4 *yds,* the last term 400 *yds,* and the number of terms 100.

But $(\overset{\text{yds.}}{4}+\overset{\text{yds.}}{400})\times 50=\overset{\text{yds.}}{20200}=\overset{\text{miles}}{11}\,..\,\overset{\text{yds.}}{840}$. Ans.

APPLICATION OF ARITHMETICAL PROGRESSION TO MILITARY AFFAIRS.

QUESTION I.

The 1st term is $\overset{\text{man}}{1}$, the com. diff. $\overset{\text{men}}{2}$, and the number of terms 30.

But $(\overset{\text{men}}{2}\times\overset{\text{men}}{29})+\overset{\text{man}}{1}=\overset{\text{men}}{59}$ the last term, or *number of men in the last rank.*

And $\frac{1+59}{2}\times 30=900$ men, STRENGTH of the BATTALION. Ans.

(Key to Vol. I. page 211.)

QUESTION II.

The 1st term is $\overset{\text{leag.}}{2}$, the com. diff. $\overset{\text{leag.}}{1\frac{1}{2}}$, and the number of terms 12.

But $(\overset{\text{leag.}}{1\frac{1}{2}}\times 11)+\overset{\text{leag.}}{2}=\overset{\text{leag.}}{18\frac{1}{2}}$ leagues, *last day's march.*

And $(\overset{\text{leag.}}{2}+\overset{\text{leag.}}{18\frac{1}{2}})\times 6$(=*half n°. terms*)$=123$ leagues, Length of the March. Ans.

QUESTION III.

The 1st term is $\overset{\text{yds.}}{15}$, the last term $\overset{\text{yds.}}{3}$, and the com. diff. $\overset{\text{yds.}}{2}$.

But $\frac{\overset{\text{yds.}}{15}-\overset{\text{yds.}}{3}+\overset{\text{yds.}}{2}}{2}=7$ the number of terms, or *nights employed.*

And $\frac{\overset{\text{yds.}}{3}+\overset{\text{yds.}}{15}}{2}\times 7=63$ yards, Length of the whole Sap. Ans.

(Page 212.)

QUESTION IV.

The 1st rank is $\overset{\text{gabions}}{4}$, the last rank $\overset{\text{gabions}}{9}$, and the number of ranks 6.

But $\frac{\overset{\text{gab.}}{9}-\overset{\text{gab.}}{4}}{6-1}=1$ *the common difference.*

And $(4+9)\times 3=39$ the Number of Gabions. Ans.

QUESTION V.

If the straight line joining the two Camps pass through the Post in question, it is evident that $18\frac{1}{2}$ leagues is *the Sum of the series* of either detachment.

Wherefore, for the 1st Detachment,

$$\begin{cases} \overset{\text{leag.}}{(18\tfrac{1}{2}\times 2)}\div 5\,(=\textit{No. of terms})=\overset{\text{leag.}}{7\tfrac{2}{5}} \text{ the sum of the Extremes.} \\ (1\tfrac{1}{2}\times 5)-1\tfrac{1}{2}=6 \text{ the difference of the Extremes.} \end{cases}$$

Hence (*by ½ sum and ½ diff. &c.*) $\overset{\text{leag.}}{\tfrac{7}{10}}$, $\overset{\text{leag.}}{2\tfrac{2}{10}}$, $\overset{\text{leag.}}{3\tfrac{7}{10}}$, $\overset{\text{leag.}}{5\tfrac{2}{10}}$, $\overset{\text{leag.}}{6\tfrac{7}{10}}$ the SERIES.

For the 2d Detachment,

$$\begin{cases} \overset{\text{leag.}}{(18\tfrac{1}{2}\times 2)}\div 4\,(=\textit{No. of terms})=\overset{\text{leag.}}{9\tfrac{1}{4}} \text{ the sum of the Extremes.} \\ (2\ \times 4)-2=6 \text{ the difference of the Extremes.} \end{cases}$$

Hence (*by half sum and half diff. &c.*) $\overset{\text{leag.}}{1\tfrac{5}{8}}$, $\overset{\text{leag.}}{3\tfrac{5}{8}}$, $\overset{\text{leag.}}{5\tfrac{5}{8}}$, $\overset{\text{leag.}}{7\tfrac{5}{8}}$ the SERIES.

⁂ *But as this* POST *may be any where in a line passing at right angles through the middle of the right line joining the two Camps, the Question admits of an Infinity of Answers.*

QUESTION VI.

It is manifest that the sum of the series of the detachment must be some multiple of 8, and at the same time equal to 8 times the number of terms; therefore, constructing the Series, it will be,

Days	1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	&c.
Series in Leagues,	2,	5,	8,	11,	14,	17,	20,	23,	&c.
Daily *Sum of the Series,*	2,	7,	15,	26,	40,	57,	77,	100,	&c.
Daily *Progress of the Deserter,*	8,	16,	24,	32,	40,	48,	56,	64,	&c.

By which it appears, that no term except the 5th can answer the conditions given.

Consequently 5 days, and 40 leagues. Ans.

ALGEBRAICALLY.

The 1st term is 2, the com. diff. 3, and if x be put for the number of terms, it will be, $3(x-1)+2=3x-1$ *the last term* of the SERIES.

And, $\frac{2+3x-1}{2}\times x=\frac{3x+1}{2}\times x=\frac{3x^2+x}{2}$ *the Sum* of the SERIES.

(Key to Vol. I. page 212.)

But (*by the Quest.*) $\frac{3x^2+x}{2}=8x$, that is, $3x^2+x=16x$, or, (*dividing by* x) $3x+1=16$, hence $x=5$ *days.*

Consequently, as before, 5 days, and 40 leagues. Ans.

(Page 213.)

QUESTION VII.

The 1st term is $\frac{1}{2}$, the last $9\frac{1}{2}$, and the number of terms 7.

leag. leag. leag.

Also $\frac{9\frac{1}{2}-\frac{1}{2}}{7-1}=1\frac{1}{2}$ *the common difference.*

Consequently,

Series in Leagues.	The Convoy had advanced in Leagues.	*Days.*	The Escort had advanced in Leagues.	Dist. of the Escort from the Convoy, at the end of each March. Leagues.	*Nights.*
$\frac{1}{2}$	5	1	$\frac{1}{2}$	$4\frac{1}{2}$	1st
2	10	2	$2\frac{1}{2}$	$7\frac{1}{2}$	2d
$3\frac{1}{2}$	15	3	6	9	3d
5	20	4	11	9	4th
$6\frac{1}{2}$	25	5	$17\frac{1}{2}$	$7\frac{1}{2}$	5th
8	30	6	$25\frac{1}{2}$	$4\frac{1}{2}$	6th
$9\frac{1}{2}$	35	7	35	0	7th

Whereby the Answer is obvious.

GEOMETRICAL PROPORTION.

(Page 220.)

Ex. 3. $3^{11}=177147$

And $\frac{(3\times177147)-1}{3-1}=265720$. Ans.

Ex. 4. $(\frac{1}{3})^{11}=\frac{1}{177147}$.

And $1-(\frac{1}{3}\times\frac{1}{177147})=\frac{531440}{531441}$.

Lastly $\frac{531440}{531441}\div\frac{2}{3}=\frac{1594320}{1062882}=\frac{265720}{177147}$. Ans.

(Key to Vol. I. page 220.)

Ex. 5. $2^{99} = 2^{9}\times 2\times 2^{10}\times 2^{20}\times 2^{40}\times 2^{10}\times 2^{9}.$

$$
\begin{array}{lr}
\text{But } 2^{9} = & 512 \\
2^{1} & 2 \\
\hline
2^{10} = & 1024 \\
2^{10} & 1024 \\
\hline
2^{20} = & 1048576 \\
2^{20} & 1048576 \\
\hline
2^{40} = & 1099511627776 \\
2^{40} & 1099511627776 \\
\hline
2^{80} = & 1208925819614629174706176 \\
2^{10} & 1024 \\
\hline
2^{90} = & 1237940039285380274899124224 \\
2^{9} & 512 \\
\hline
2^{99} = & 633825300114114700748351602688 \\
\times^{\text{ly}}\ \textit{by the}\ \text{First term} & 1 \\
\hline
 & 633825300114114700748351602688 \\
\times^{\text{ly}}\ \textit{by the}\ \text{Ratio} & 2 \\
\hline
 & 1267650600228229401496703205376
\end{array}
$$

Subtract the First term (1) and *divide by* the diff. between the 1st term and ratio, (1).

The Quotient is 1267650600228229401496703205375 Sum of the SERIES.

SIMPLE EQUATIONS.

(Page 226.)

Ex 1.

$$
\begin{array}{r|l}
\text{Given.} & 2x-5+16=21 \\
\text{Collect.} & 2x \quad +11=21 \\
\text{Transp. 11.} & 2x \qquad =21-11=10 \\
\text{Divide by 2.} & x \qquad = \qquad 5 \ \ \text{Ans.}
\end{array}
$$

(Key to Vol. I. page 226.)

Ex. 2.	Given.	$6x-15=x+6$
	From both sides take x.	$5x-15=6$
	Transp. -15.	$5x=6+15=21$
	Divide by 5.	$x=\frac{21}{5}=4\frac{1}{5}$ Ans.

Ex. 3.	Given.	$8-3x+12=30-5x+4$
	From both sides take $-3x+4$.	$8+8=30-2x$
	Collect and transp. $-2x$.	$2x+16=30$
	Transp. 16.	$2x=30-16=14$
	Divide by 2.	$x=7$ Ans.

Ex. 4.	Given.	$x+\frac{x}{3}-\frac{x}{4}=13$
	$\times^{\text{ly}}$ by 3.	$3x+x-\frac{3x}{4}=39$
	$\times^{\text{ly}}$ by 4.	$12x+4x-3x=156$
	Collect.	$13x=156$
	Divide by 13.	$x=12$ Ans.

Ex. 5.	Given.	$3x+\frac{x}{2}+2=5x-4$
	Take $3x$ from both sides.	$\frac{x}{2}+2=2x-4$
	Transp. -4.	$\frac{x}{2}+6=2x$
	$\times^{\text{ply}}$ by 2.	$x+12=4x$
	From both sides take x.	$12=3x$
	Divide by 3.	$4=x$, or $x=4$ Ans.

Ex. 6.	Given.	$4ax+\frac{a}{3}-2=ax-bx$
	$\times^{\text{ly}}$ by 3.	$12ax+a-6=3ax-3bx$
	From both sides take $3ax$.	$9ax+a-6=-3bx$
	Transp. $-3bx$, and $a-6$.	$9ax+3bx=6-a$
	Divide by $9a+3b$.	$x=\frac{6-a}{9a+3b}$ Ans.

(Key to Vol. I. page 226.)

Ex. 7.

Given.	$\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=\frac{1}{2}$
$\times^{\text{ly}}$ by 3.	$x-\frac{3x}{4}+\frac{3x}{5}=\frac{3}{2}$
$\times^{\text{ly}}$ by 4.	$4x-3x+\frac{12x}{5}=6$
Collect, and $\times^{\text{ly}}$ by 5.	$5x+12x=30$
Collect and divide by 17.	$x=\frac{30}{17}=1\frac{13}{17}$. Ans.

Ex. 8.

Given.	$\sqrt{4+x}=4-\sqrt{x}$
Square both sides.	$4+x=16-8\sqrt{x}+x$
From both sides take x.	$4=16-8\sqrt{x}$
Transp. $-8\sqrt{x}$, and 4.	$8\sqrt{x}=16-4=12$
Square both sides.	$64x=144$
Divide by 64.	$x=2\frac{1}{4}$ Ans.

Ex. 9.

Given.	$4a+x=\frac{x^2}{4a+x}$
$\times^{\text{ly}}$ by $4a+x$.	$16a^2+8ax+x^2=x^2$
Take x^2 from both sides, and transp. $16a^2$.	$8ax=-16a^2$
Divide by $8a$.	$x=-2a$. Ans.

Ex. 10.

Given.	$\sqrt{4a^2+x^2}=\sqrt[4]{4b^4+x^4}$
Square both sides.	$4a^2+x^2=\sqrt[2]{4b^4+x^4}$
Square both sides.	$16a^4+8a^2x^2+x^4=4b^4+x^4$
From both sides take x^4.	$16a^4+8a^2x^2=4b^4$
Transp. $16a^4$.	$8a^2x^2=4b^4-16a^4$
Divide by 4.	$2a^2x^2=b^4-4a^4$
Divide by $2a^2$.	$x^2=\frac{b^4-4a^4}{2a^2}$
Extract the Sq. Root on both sides.	$x=\sqrt{\frac{b^4-4a^4}{2a^2}}$. Ans.

(Key to Vol. I. page 226.)

Ex. 11.	Given.	$\sqrt{x}+\sqrt{2a+x} = \frac{4a}{\sqrt{2a+x}}$
	$\times^{ly}$ by $\sqrt{2a+x}$.	$\sqrt{2ax+x^2}+2a+x = 4a$
	Transp. $2a+x$.	$\sqrt{2ax+x^2} = 4a-2a-x = 2a-x$
	Square both sides.	$2ax+x^2 = 4a^2-4ax+x^2$
	Cancel x^2.	$2ax = 4a^2-4ax$
	Transp. $-4ax$.	$6ax = 4a^2$
	Divide by $6a$.	$x = \frac{4a^2}{6a} = \frac{2}{3}a$. Ans.

Ex. 12.	Given.	$\frac{a}{1+2x}+\frac{a}{1-2x} = 2b$
	Reduce the fractions.	$\frac{2a}{1-4x^2} = 2b$
	$\times^{ly}$ by $1-4x^2$.	$2a = 2b-8bx^2$
	Transpose $-8bx^2$, and $2a$.	$8bx^2 = 2b-2a$
	Divide by $8b$.	$x^2 = \frac{b-a}{4b} = \frac{1}{4}(\frac{b-a}{b})$
	Extr. the Square Root on both sides.	$x = \frac{1}{2}\sqrt{\frac{b-a}{b}}$. Ans.

Ex. 13.	Given.	$a+x = \sqrt{a^2+x\sqrt{4b^2+x^2}}$
	Square both sides.	$a^2+2ax+x^2 = a^2+x\sqrt{4b^2+x^2}$
	Cancel a^2, and divide by x.	$2a+x = \sqrt{4b^2+x^2}$
	Square both sides.	$4a^2+4ax+x^2 = 4b^2+x^2$
	Cancel x^2.	$4a^2+4ax = 4b^2$
	Transp. $4a^2$, and div. by 4.	$ax = b^2-a^2$
	Divide by a.	$x = \frac{b^2}{a}-a$. Ans.

(Page 228.)

To exterminate Two Unknown Quantities.

RULE. I.

Ex. 2.

Given* $\left\{\begin{array}{l} \frac{x}{2}+2y=a \\ \frac{x}{2}-2y=b \end{array}\right\}$

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* *This Equation belongs more properly to* Rule III, *admitting there of a much Shorter and more Elegant Solution.*

(Key to Vol. I. page 228.)

In the 1st equat. transp. $2y$, and $\times^{\text{ly}}$ by 2.	$x=2a-4y$
In the 2d equat. transp. $-2y$, and $\times^{\text{ly}}$ by 2.	$x=2b+4y$
Put these values of x equal to one another.	$2a-4y=2b+4y$
Transpose $-4y$ and $2b$.	$2a-2b=8y$
Divide by 8.	$\frac{a-b}{4}=y$

But $x=2a-4y=2a-(a-b)=a+b$, and the Answer *obvious*.

Ex. 3. Given.* $\left\{\begin{array}{l}3x+y=22\\ x+3y=18\end{array}\right\}$

In the first equat. transpose $3x$.	$y=22-3x$
In the 2d equat. tr. x, and divide by 3.	$y=\frac{18-x}{3}$
Put these values of y equal to one another.	$22-3x=\frac{18-x}{3}$
Multiply by 3.	$66-9x=18-x$
Cancel $-x$, and transpose.	$66-18=8x$
Collect and divide by 8.	$6=x$

But $y=22-3x=22-18=4$.

Wherefore $x=6$, and $y=4$. Ans.

Ex. 4.† Given $\left\{\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=4\\ \frac{1}{3}x+\frac{1}{2}y=3\frac{1}{2}\end{array}\right\}$

In the 1st equat. tr. $\frac{1}{2}x$, and $\times^{\text{ly}}$ by 3.	$y=12-\frac{3x}{2}$
In the 2d equat. tr. $\frac{1}{3}x$, and $\times^{\text{ly}}$ by 2.	$y=7-\frac{2x}{3}$
Put these values of y equal to one another.	$12-\frac{3x}{2}=7-\frac{2x}{3}$
Take 7 from both sides.	$5-\frac{3x}{2}=-\frac{2}{3}x$
Multiply by 2, and by 3.	$30-9x=-4x$
Cancel $-4x$ on both sides, and tr.	$30=5x$
Divide by 5.	$6=x$

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* *The Observation made on the last Equation is equally applicable to the present.*

† *This Example and the next are likewise Equations in the Third Rule.*

(Key to Vol. I. page 228.)

But $y=12-\frac{3x}{2}=12-\frac{18}{2}=12-9=3$, and the Ans. *obvious.*

Ex. 5.

$$\text{Given}\left\{\begin{array}{l}\frac{2x}{3}+\frac{3y}{5}=\frac{22}{5}\\ \frac{3x}{5}+\frac{2y}{3}=\frac{67}{15}\end{array}\right\}$$

In the 1st equat. tr. $\frac{2x}{3}$, and $\times^{ly}$ by 5.	$3y=22-\frac{10x}{3}$
Divide by 3.	$y=\frac{22}{3}-\frac{10x}{9}$
In the 2d equat. tr. $\frac{3x}{5}$, and div. by $\frac{2}{3}$.	$y=\frac{67}{10}-\frac{9}{10}x$
Put these values of y equal to one another.	$\frac{67}{10}-\frac{9}{10}x=\frac{22}{3}-\frac{10x}{9}$
Reduce.	$603-81x=660-100x$
Cancel $81x$ on both sides.	$603=660-19x$
Take 603 from both sides.	$0=57-19x$
Transpose, and divide by 19.	$x=3$

But $y=\frac{22}{3}-\frac{10x}{9}=\frac{22}{3}-\frac{30}{9}=\frac{22}{3}-\frac{10}{3}=\frac{12}{3}=4$, and the Ans. *obvious.*

Ex. 6.

$$\text{Given}\left\{\begin{array}{l}x+2y=s\\ x^2-4y^2=d^2\end{array}\right\}$$

In the 1st equat. tr. $2y$.	$x=s-2y$
In the 2d equat. tr. $-4y^2$, and extr. $\surd$.	$x=\sqrt{d^2+4y^2}$
Put these values of x equal to one another, and Square.	$s^2-4sy+4y^2=d^2+4y^2$
Cancel $4y^2$.	$s^2-4sy=d^2$
Transp. $-4sy$, and d^2.	$s^2-d^2=4sy$
Divide by $4s$.	$\frac{s^2-d^2}{4s}=y$
But $x+2y=s$, that is,	$x+\frac{s^2-d^2}{2s}=s$

Transpose $\frac{s^2-d^2}{2s}$	$x=s-\frac{s^2-d^2}{2s}=\frac{2s^2-s^2+d^2}{2s}=\frac{s^2+d^2}{2s}.$

Therefore $x=\frac{s^2+d^2}{2s}$, and $y=\frac{s^2-d^2}{4s}$. Ans.

(Key to Vol. I. page 228.)

Ex. 7.

$$\text{Given}\left\{\begin{array}{l} x-2y=d \\ x:y::a:b \end{array}\right\}$$

By the 1st equat. | $x=d+2y$

By the ratio given. | $x=\frac{ay}{b}$

Therefore (*Eucl.* i. *Ax.* 1.) | $\frac{ay}{b}=d+2y$

×ly by b. | $ay=bd+2by$

Transp. $2by$. | $ay-2by=bd$

Divide by $a-2b$. | $y=\frac{bd}{a-2b}$

$$\text{But } x=d+2y=d+\frac{2bd}{a-2b}=\frac{ad-2bd+2bd}{a-2b}=\frac{ad}{a-2b}.$$

Wherefore the values of x and y are known.

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RULE II.

Ex. 2.

$$\text{Given}\left\{\begin{array}{l} 2x+3y=29 \\ 3x-2y=11 \end{array}\right\}$$

In the 1st equat. | $x=\frac{29-3y}{2}$

Subst. this val. of x for x in the 2d equat. | $\frac{87-9y}{2}-2y=11$

×ly by 2, collect, tr. and div. by 13. | $5=y$

Wherefore, by step 1st. | $\frac{29-15}{2}=7=x$

$$\left.\begin{array}{r} 5=y \\ \frac{29-15}{2}=7=x \end{array}\right\}\text{Ans.}$$

Ex 3.

$$\text{Given*}\left\{\begin{array}{l} x+y=14 \\ x-y=\ \ 2 \end{array}\right\}$$

In the 1st equation. | $x=14-y$

Subst. this val. of x for x in the 2d equat. | $14-y-y=2$

Tr. collect, and divide by 2. | $6=y$

And, by step 1st. | $8=x$

$$\left.\begin{array}{l} 6=y \\ 8=x \end{array}\right\}\text{Ans.}$$

---

* *This Equation solved by the* Rule *is neither a short nor elegant Operation.* For

Add the given Equations, and divide by 2. | $x=8$

Subtract the 2d Equat. from the 1st, and div. by 2. | $y=6$

$$\left.\begin{array}{l} x=8 \\ y=6 \end{array}\right\}\text{Ans.}$$

(Key to Vol. I. page 230.)

Ex. 4.

$$\text{Given}\left\{\begin{array}{l} x : y :: 3 : 2 \\ x^2 - y^2 = 20 \end{array}\right\}$$

By the given ratio.	$x = \frac{3y}{2}$
Square both sides.	$x^2 = \frac{9y^2}{4}$
Subst. this val. of x^2 for x^2 in the given equat.	$\frac{9y^2}{4} - y^2 = 20$
$\times^{ly}$ by 4, and collect.	$5y^2 = 80$
Divide by 5, and extr. $\surd$.	$y = 4$
But, by the Quest.	$x : y, \text{or } 4 :: 3 : 2$
Wherefore,	$x = 6$

Ex. 5.

$$\text{Given}\left\{\begin{array}{l} \frac{x}{3} + 3y = 21 \\ \frac{y}{3} + 3x = 29 \end{array}\right\}$$

In the 1st equat. tr. $3y$, and $\times^{ly}$ by 3.	$x = 63 - 9y$
Subst. this val. of x for x in the 2d equat.	$\frac{y}{3} + 189 - 27y = 29$
$\times^{ly}$ by 3, tr. and collect.	$480 = 80y$
Divide by 80.	$6 = y$
But, by the 2d equat. $\frac{6}{3} + 3x = 29$, or	$9 = x$

$$\left.\begin{array}{l} 6 = y \\ 9 = x \end{array}\right\}\text{Ans.}$$

Ex. 6.

$$\text{Given}\left\{\begin{array}{l} 10 - \frac{x}{2} = \frac{y}{3} + 4 \\ \frac{x-y}{2} + \frac{x}{4} - 2 = \frac{3y-x}{5} - 1 \end{array}\right\}$$

In the 1st equat. cancel 4.	$6 - \frac{x}{2} = \frac{y}{3}$
Transp. and $\times^{ly}$ by 2.	$12 - \frac{2y}{3} = x$
Reduce the 2d equat.	$19x = 22y + 20$
For x subst. $12 - \frac{2}{3}y$.	$228 - \frac{38y}{3} = 22y + 20$
$\times^{ly}$ by 3.	$684 - 38y = 66y + 60$
Transp. and div. by 104.	$6 = y$
But (by step. 2d).	$12 - \frac{2}{3}y$, that is, $12 - 4 = 8 = x$

$$\left.\begin{array}{l} 6 = y \\ 12 - 4 = 8 = x \end{array}\right\}\text{Ans.}$$

(Key to Vol. I. page 230.)

Ex. 7. Given $\left\{\begin{matrix} x:y::4:3 \\ x^3-y^3=37 \end{matrix}\right\}$

By the given ratio.	$x=\frac{4y}{3}$	
Cube both sides.	$x^3=\frac{64y^3}{27}$	
Subst. this val. of x^3 for x^3 in the given equat.	$\frac{64y^3}{27}-y^3=37$	
×$^{\text{ly}}$ by 27.	$64y^3-27y^3=999$	
Collect and divide by 37.	$y^3=27$	
Extr. $\sqrt[3]{}$ on both sides.	$y=3$	Ans.
But, by step 1st.	$x=\frac{4y}{3}=\frac{12}{3}=4$	

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RULE III.

Ex. 1. Given $\left\{\begin{matrix} \frac{x+8}{4}+6y=21 \\ 5x+\frac{y+6}{3}=23 \end{matrix}\right\}$

Reduce the 1st equat.	$x+24y=76$	
Reduce the 2d equat.	$15x+y=63$	
×$^{\text{ly}}$ the 1st equat. red. by 15.	$15x+360y=1140$	
From this equat. subtr. the 2d reduced.	$359y=1077$	
Divide by 359.	$y=3$	Ans.
But, by step. 1st (subst$^{\text{g}}$. 3 for y.)	$x+72=76$, or $x=4$)	

Ex. 2. Given $\left\{\begin{matrix} \frac{3x-y}{4}+10=13 \\ \frac{3y+x}{2}+5=12 \end{matrix}\right\}$

Reduce the 1st equation	$3x-y=12$
Reduce the 2d equation	$x+3y=14$

(Key to Vol. I. page 231.)

$\times^{\text{ly}}$ the 2d equat. reduced, by 3.	$3x+9y=42$	
From this equat. take the 1st reduced.	$10y=30$	
Divide by 10.	$y=3$	Ans.
But, by step. 2d, subst$^{\text{s}}$. 3 for y.	$x+9=14$, or, $x=5$	

Ex. 3.

$$\text{Given}\left\{\begin{array}{l}\dfrac{3x+4y}{5}+\dfrac{x}{4}=10\\[2ex] \dfrac{6x-2y}{3}+\dfrac{y}{6}=14\end{array}\right\}$$

Reduce the 1st equat.	$17x+16y=200$	
Reduce the 2d equat.	$12x-3y=84$	
$\times^{\text{ly}}$ the 1st equat. red. by 3.	$51x+48y=600$	
$\times^{\text{ly}}$ the 2d equat. red. by 16.	$192x-48y=1344$	
Add together the 2 last steps.	$243x=1944$	
Divide by 243.	$x=8$	Ans.
But (by the 2d equat. reduced) substituting 8 for x.	$96-3y=84$, or $y=4$	

Ex. 4.

$$\text{Given}\left\{\begin{array}{l}3x+4y=38\\ 4x-3y=9\end{array}\right\}$$

$\times^{\text{ly}}$ the 1st equat. by 4.	$12x+16y=152$	
$\times^{\text{ly}}$ the 2d equat. by 3.	$12x-9y=27$	
Subtr. the 2d step from the 1st.	$25y=125$	
Divide by 25.	$y=5$	Ans.
By the question, subst$^{\text{s}}$. 5 for y.	$3x+20=38$, or $x=6$	

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To exterminate Three or more Unknown Quantities.

Ex. 2.

$$\text{Given}\left\{\begin{array}{l}x+y+z=18\\ x+3y+2z=38\\ x+\frac{1}{3}y+\frac{1}{2}z=10\end{array}\right\}\text{or}\left\{\begin{array}{l}x=18-y-z\\ x=38-3y-2z\\ x=10-\frac{1}{3}y-\frac{1}{2}z\end{array}\right\}$$

Putting the 1st equat. equal to the 2d, and reducing.	$2y+z=20$	
$\times^{\text{ly}}$ by 2.	$4y+2z=40$	
Putting the 1st equat. equal to the 3d, and reducing.	$4y+3z=48$	
By Subtraction.	$z=8$	
But (by step 2d) substituting 8 for z,	$2y+8=20$, or $y=6$	Ans.
And, subst$^{\text{s}}$. 6 for y and 8 for z,	$x+14=18$, or $x=4$	

(Key to Vol. I. page 233.)

Ex. 3.

$$\text{Given}\left\{\begin{array}{l} x+\frac{1}{2}y+\frac{1}{3}z=27 \\ x+\frac{1}{3}y+\frac{1}{4}z=20 \\ x+\frac{1}{4}y+\frac{1}{5}z=16 \end{array}\right\}\text{or}\left\{\begin{array}{l} x=27-\frac{1}{2}y-\frac{1}{3}z \\ x=20-\frac{1}{3}y-\frac{1}{4}z \\ x=16-\frac{1}{4}y-\frac{1}{5}z \end{array}\right\}$$

Putting the 1st equat. equal to the 2d.	$2y+z=84$
Putting the 1st equat. equal to the 3d.	$15y+8z=660$
× ly the 1st step by 8.	$16y+8z=672$
From this equat. subtr. the 2d step.	$y=12$
But, by step 1st $2y+z=84$, hence	$z=60$
And, subtg. 12 for y, and 60 for z &c.	$x=1$

$$\left.\begin{array}{l} y=12 \\ z=60 \\ x=1 \end{array}\right\}\text{Ans.}$$

Ex. 4.

$$\text{Given}\left\{\begin{array}{l} x-y=2 \\ x-z=3 \\ y+z=9 \end{array}\right\}$$

Subtr. the 1st equat. from the 2d.	$y-z=1$
Add this equat. to the 3d.	$2y=10$, or $y=5$
Subtr. the 1st step from the 3d equat.	$2z=8$, or $z=4$
But, (by the 2d equat.)	$x-4=3$, or $x=7$

$$\left.\begin{array}{l} y=5 \\ z=4 \\ x=7 \end{array}\right\}\text{Ans.}$$

Ex. 5.

$$\text{Given}\left\{\begin{array}{l} 2x+3y+4z=34 \\ 3x+4y+5z=46 \\ 4x+6y+8z=58 \end{array}\right\}\text{or}\left\{\begin{array}{l} x=\dfrac{34-3y-4z}{2} \\ x=\dfrac{46-4y-5z}{3} \\ x=\dfrac{58-6y-8z}{4} \end{array}\right\}$$

Putting the 1st equat. equal to the 2d. | $y+2z=10$.

But, on comparing the 1st and 3d equations, an absurdity appears, and therefore no three numbers can answer the conditions.

If, however, in the last of the given Equations $5y$ be intended, and if 58 be a mistake for 66, then, $x=6$, $y=2$, and $z=4$.

NOTE.—*In some Editions of the Course the last Equation is read* $4x+5y+6z=58$.

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QUESTIONS FOR PRACTICE.

Quest. 1. Let x represent the greater number, and y the less.

$$\text{By the Quest.}\left\{\begin{array}{l} x-y=4 \\ x^2-y^2=64 \end{array}\right\}\text{or}\left\{\begin{array}{l} x=4+y \\ x^2=64+y^2 \end{array}\right\}$$

Square both sides of the 1st equation transposed.	$x^2=y^2+8y+16$
Compare the two values of x^2.	$y^2+8y+16=64+y^2$
Cancel y^2.	$8y+16=64$
Transpose 16, and divide by 8.	$y=6$
But $x-y=4$, that is,	$x=10$

Consequently 10 *and* 6 *are the numbers sought.*

Quest. 2. If x be put for the 1st number, and y for the 2d, it is,

$$\text{By the Quest. }\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=9\\ \frac{x}{4}+\frac{y}{5}=5\end{array}\right\}$$

Reduce the 1st. equat.	$3x+2y=54$	
Reduce the 2d equat.	$5x+4y=100$	
$\times^{\text{ly}}$ the 1st equat. reduced, by 2.	$6x+4y=108$	
From this equat. take the 2d equat. reduced.	$x=8$	*The two numbers.*
And, because by the 1st equat. reduced, $(24+2y=54.)$	$y=15$	

Quest. 3. Put x for one of the parts, and y for the other.

$$\text{By the Quest. }\left\{\begin{array}{l}x+y=20\\ \frac{x}{3}+\frac{y}{5}=6\end{array}\right\}$$

Reduce the 2d equat.	$5x+3y=90$
$\times^{\text{ly}}$ the 1st equation by 5.	$5x+5y=100$
From this equat. take the 2d equat. reduced.	$2y=10$, or $y=5$
And, by the 1st equation,	$20-y(=20-5)=x=15$

Wherefore 15 *and* 5 *are the parts required.*

Quest. 4. Put x for the 1st number, y for the 2d, and z for the 3d.

$$\text{By the Quest. }\left\{\begin{array}{l}x+y=7\\ x+z=8\\ y+z=9\end{array}\right\}$$

Subtr. the 1st equat. from the 2d.	$z-y=1$
Add this rem. to the 3d equation.	$2z=10$, or, $z=5$
Subtr. the 1st step from the 3d equat.	$2y=8$, or, $y=4$
But $x+4=7$, hence	$x=3$

Ans.

(Key to Vol. I. page 236.)

Quest. 5.* Let x represent the son's portion, and y that of the daughter.

By the Quest. $\left\{\begin{array}{l} x+y=\pounds2800 \\ x:y::5:2 \end{array}\right\}$ or $\left\{\begin{array}{l} x=\pounds2800-y \\ 2x=5y \end{array}\right\}$

Mul. the 1st equat. transposed, by 2.	$2x=\pounds5600-2y$
From this equat. subtr. the 2d equat.	$0=5600-7y$
Transp. $7y$, and divide by 7.	$y=800$
Therefore, (by the 1st equat.)	$x=2000$

$\left.\begin{array}{l} y=800 \\ x=2000 \end{array}\right\}$ Ans.

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* In common Arithmetic this Question admits of the easiest solution, for

$2+5)\pounds2800(\pounds400$ which $\times \left\{\begin{array}{l} 2=\pounds800 \text{ the daughter's portion.} \\ 5=2000 \text{ the son's portion.} \end{array}\right.$

$\pounds2800$

Quest. 6.

Given $\left\{\begin{array}{l} A+B+C=\pounds400 \\ 2A+\pounds20=B \\ A+B=C \end{array}\right\}$

Substitute B's and C's contr. in terms of A's.	$A+(2A+\pounds20)+(A+2A+\pounds20)=\pounds400$
Collect.	$6A+\pounds40=\pounds400$
Tr. and divide by 6.	$A=60$
Therefore	$B=140$
And	$C=200$

$\left.\begin{array}{l} A=60 \\ B=140 \\ C=200 \end{array}\right\}$ Ans.

Otherwise.

It is manifest, by the 3d equat. given, that $C=\pounds200$

And (by comparing the 1st and 2d equat^s.) that $3A=180$

Whence it follows that $A=\pounds60$, $B=\pounds140$, *and* $C=\pounds200$ *as before.*

(Page 237.)

Quest. 7. Put x for the number of half guineas, and y for the number of crowns.

By the Quest.	$x+y=202$
And since ($\pounds100=4000$ sixpences)	$21x+10y=4000$
$\times^{ly}$ the 1st equat. by 21.	$21x+21y=4242$
From this equat. take the 2d equation.	$11y=242$
Divide by 11.	$y=22$
From 202 take 22.	$x=180$

Consequently 22 *crowns, and* 180 *half guineas.* Ans.

(Key to Vol. I. page 237.)

Quest. 8.

$$\text{By the Quest.}\begin{cases}\overset{gs.}{2B}-20=\overset{gs.}{A}+10\\ B+10=3A-30\end{cases}$$

$\times^{ly}$ the 1st equat. by 3.	$6B-\overset{gs.}{60}=3A+\overset{gs.}{30}$	
From this equat. take the 2d equat.	$5B-70=60$	
Transp. -70gs. and divide by 5.	$B=26$	Ans.
Therefore, by the 1st equat.	$A=22$	

Quest. 9. Put $\overset{s.}{x}$ for the money the tippler had at first.

	s. s.
2*s.* spent at the 1*st house* left	$x-2$
Add $x-2$, borrowed; *had*	$2x-4$
2*s.* spent at the 2*d house* left	$2x-6$
Add $2x-6$, borrowed; *had*	$4x-12$
2*s.* spent at the 3*d house* left	$4x-14$
Add $4x-14$, borrowed; *had*	$8x-28$
2*s.* spent at the 4*th house* left	$8x-30=0$

Therefore $8x=30$, that is, $x=\overset{s.}{3}\,..\,\overset{d.}{9}$. Ans.

Quest. 10. Put $\overset{s.}{x}$ for the charge made for the man, and $\overset{s.}{y}$ that made for his wife.

$$\text{By the Quest.}\begin{cases}\overset{s.}{1}=\text{the child's reckoning.}\\ 1+\frac{x}{4}=y.\ \text{the woman's reckoning.}\\ 2+\frac{x}{4}=x.\ \text{the man's reckoning.}\end{cases}$$

$\times^{ly}$ the last equat. by 4.	$8+x=4x$	
Transp. x, and divide by 3.	$\frac{8}{3}=x=\overset{s.}{2}\,..\,\overset{d.}{8}=32d.$	Ans.
And, by the 2d equat.	$y=1\,..\,8=20d.$	

Quest. 11. Let x, y, z respectively be the quantities of brandy, wine, and cider, in gallons.

$$\text{By the Quest.}\begin{cases}x+y+z=60\\ z-6=x\\ z+\frac{1}{5}x=y\end{cases}\ \text{or}\ \begin{cases}x=60-y-z\\ x=z-6\\ x=5y-5z\end{cases}$$

(Key to Vol. I. page 237.)

Putting the 1st equat. equal to the 2d.	$y+2z=66$
Putting the 1st equat. equal to the 3d.	$6y-4z=60$
×ing the 1st result by 2.	$2y+4z=132$
To this adding the 2d result, and dividing by 8.	$y=24$
By the 1st result.	$z=21$
And by the 1st equation.	$x=15$

Wherefore 15 *galls. Brandy,* 24 *galls. Wine, and* 21 *galls. Cider.* Ans.

Quest. 12. Put x for the number of troops in the whole army, z for the side of the 1st square, and $z+1$ for the side of the 2d square.

Then, by the Question. $\begin{cases} z^2+284=x. \\ z^2+2z+1-25=x. \end{cases}$

Therefore (Eucl. i. Ax. 1.) $z^2+284=z^2+2z-24$.

Cancel z^2, and reduce.	$154=z$
But	$154^2+284=x=24000$ *men.* Ans.

Quest. 13. Let x represent the number sought.

Then, by the Question.	$x+3:x+5::x+5:x+8$
×ing extremes and means.	$x^2+11x+24=x^2+10x+25$
Cancel $x^2+10x+24$.	$x=1$ *the number sought.*

Quest. 14. For the three traders put A, B, and C.

By the Quest. $\left\{\begin{array}{l} A+B+C=£760. \\ A+B-£240=C. \\ B+C-£360=A. \end{array}\right\}$ or, $\left\{\begin{array}{l} A=£760-B-C. \\ A=£240-B+C. \\ A=B+C-£360. \end{array}\right.$

Comparing the 1st and 2d equations.	$£520=2C$, that is, $C=£260$.
Comparing the 1st and 3d equations.	$£1120=2C+2B$, that is, $B=£300$.
By the 1st equat. therefore,	$£200=A$.

Consequently, the 1*st* £200, the 2*d* £300, *and* the 3*d* £260. Ans.

(Page 238.)

Quest. 15. Let x and y represent the two numbers sought.

By the Question $x:y::3:4$, hence $4x=3y$, or, $12x=9y$.
Also $xy=12x+12y$.

(Key to Vol. I. page 238.)

For $12x$ substitute $9y$.	$xy=9y+12y=21y$.
Divide by y.	$x=21$.
But	$x:y::3:4::21:28$.

Therefore 21 *and* 28 *are the numbers required.*

Quest 16. Put $\overset{s.}{x}$ for the reckoning, and y for the number of persons.

It is by the Question. $\left\{\frac{x}{y+4}+1=\frac{x}{y-3}-1=\frac{x}{y}\right\}$

Comparing the 1st expression with the 3d, and reducing. $\}$ $x=\frac{1}{4}(y^2+4y)$

Comparing the 2d expression with the 3d, and reducing. $\}$ $x=\frac{1}{3}(y^2-3y)$

Therefore (Eucl. i. Ax. 1.)	$3y^2+12y=4y^2-12y$.
Reduce.	$24y=y^2$.
Divide by y.	$24=y=$ *the numb. of persons.*

But $\frac{x}{28}+1$(substituting 24 for y)$=\frac{x}{24}$, that is, $24x+672=28x$, or $x=168$ *shillings.*

Now 168*s.*=8 guineas, and $\frac{168s.}{24}=7$ *shillings for each person.*

Therefore

24 *persons,* 7*s. paid by each,* and 8 *guineas the whole reckoning.* Ans.

Quest. 17. Let x represent the 1st horse, and y the 2d.

Then, by the Quest. $\begin{cases} x+18=2y+6, \text{ that is, } x=2y-12. \\ 3x+9=y+18. \end{cases}$

Substitute $2y-12$ for x.	$6y-27=y+18$.
Reduce.	$5y=45$, or, $y=9$.
But	$x=2y-12=18-12=6$.

Therefore the 1*st horse was worth* £6, *and the* 2*d horse* £9. Ans.

Quest. 18. Let x and y represent the two numbers;

By the Question. $\begin{cases} x:y::2:3, \text{ that is, } 3x=2y. \\ x+6:y+6::4:5, \text{ that is, } 5x+6=4y. \end{cases}$

From double the 1st equat. take the 2d.	$x-6=0$, or, $x=6$.
But, by the 1st equation	$3x=2y=18$, hence $y=9$.

The two numbers therefore are 6 *and* 9.

(Key to Vol. I. page 238.)

Quest. 19. Put x and y for the two numbers sought.

By the Question $\left\{\begin{array}{l} x:y::x+y:20, \\ x:y::x-y:10, \end{array}\right\}$ or, $\left\{\begin{array}{l} 20x=xy+y^2 \\ 10x=xy-y^2 \end{array}\right\}$

Add the 1st equat. to the 2d, and div. by x.	$30=2y$, or, $y=15$
And by the 1st equat. substituting 15 for y.	$20x=15x+225$
Reduce.	$5x=225$, and $x=45$.

Therefore 15 *and* 45 *are the numbers required.*

Quest. 20. If x be put for the greater number, and y for the less.

By the Question. $\left\{\begin{array}{l} x-y:x+y::2:3 \\ x+y:xy\quad::3:5 \end{array}\right\}$ or, $\left\{\begin{array}{l} 3x-3y=2x+2y \\ 5x+5y=3xy \end{array}\right\}$

Reduce the 1st equation.	$x=5y$, that is, $x-5y=0$.
Add the 2d equat. to the 3d.	$6x=3xy$
Divide by x.	$6\ =3y$, hence $y=2$.
But	$x=5y=10$.

Therefore 2 *and* 10 *are the Answer.*

Quest. 21. Let x, y, and z represent the three numbers.

By the Question. $\left\{\begin{array}{l} x:z::5:9 \\ x,\ y,\ z \textit{ have equal diff.} \\ x+y+z=63 \end{array}\right\}$ or, $\left\{\begin{array}{l} x=\frac{5}{9}z. \\ x=2y-z. \\ x=63-y-z. \end{array}\right\}$

Comparing the 1st and 2d equations, and reducing.	$14z=18y$.
Comparing the 1st and 3d equations, and reducing.	$14z=567-9y$.
Therefore	$18y=567-9y$, that is, $y=21$.

But $14z=18y$, consequently $z=27$, and $x=\frac{5}{9}z=15$.

Hence 15, 21, *and* 27 *are the three numbers required.*

Quest. 22. Put x for the greater part, and y for the less;

By the Question. $\left\{\begin{array}{l} x+y\ =\ 24 \\ \frac{x}{y}:\frac{y}{x}::4:1 \end{array}\right\}$ or, $\left\{\begin{array}{l} x=24-y. \\ \frac{x}{y}=\frac{4y}{x}. \end{array}\right\}$

(Key to Vol. I. page 238.)

Substituting $24-y$, for x.	$\frac{24-y}{y}=\frac{4y}{24-y}$
Reduce.	$576-48y+y^2=4y^2$
Extract $\surd$ on both sides.	$24-y=2y$, or $y=8$
But $x=24-y$.	that is, $x=16$.

8 and 16, therefore, are the two parts.

Quest. 23. Let x be the age of the elder, and y the age of the younger son.

By the Question. $\begin{cases} x+y+18=2x \\ x-y-6=y \end{cases}$ or, $\begin{cases} x=y+18 \\ x=2y+6. \end{cases}$

Comparing the 1st and 2d equations, and reducing, $y=12$.

But $2y+6=x=30$.

Hence 12 and 30 are the Answer.

(Page 239.)

Quest. 24. Assume x, y, z, and w, for the four numbers.

By the Question. $\begin{cases} x+y+z=13 \\ x+y+w=15 \\ x+z+w=18 \\ y+z+w=20 \end{cases}$

Take the 1st equat. from the 2d.	$w=2+z$
Subst. $2+z$ for w in the 3d equat.	$x+2z=16$
From this equat. take the 1st.	$z-y=3$, or $z=y+3$
Subst. known values in the 4th.	$y,+y+3,+2+y+3=20$
Collect.	$3y+8=20$, or $y=4$
But	$z=y+3=4+3=7$
And	$x=16-2z=16-14=2$
Lastly	$w=2+z=2+7=9$.

Therefore the numbers sought are, 2, 4, 7, and 9.

Quest. 25. Let the four parts be severally represented by x, y, w, and z.

By the Question. $\begin{cases} x+3=y-3 \\ x+3=3w \\ x+3=\frac{1}{3}z \\ x+y+z+w=48 \end{cases}$ or, $\begin{cases} x=y-6 \\ x=3w-3 \\ x=\frac{1}{3}(z-9) \\ \text{- - - - - - -} \end{cases}$

(Key to Vol. I. page 239.)

Comparing the 1st and 2d values of x.	$\frac{y}{3}-1=w$
Comparing the 1st and 3d values of x.	$3y-9=z$
In the 4th equat. given, express x, w, z, in terms of y.	$y-6+y+\frac{y}{3}-1+3y-9=48.$
Collect and reduce.	$y=12.$

But	$y-6=x=12-6=\ 6.$
And	$\frac{1}{3}y-1=w=\ 4-1=\ 3.$
Also	$3y-9=z=36-9=27.$
Therefore	6, 12, 3, and 27 *are the parts sought.*

QUADRATIC EQUATIONS.

(Page 243.)

Ex. 1.	Given.	$x^2-6x\quad=\ 40$
	Complete the square.	$x^2-6x+9=\ 49$
	Extract the square root.	$x\ -3\quad=\pm 7$
	Transpose.	$x\ =3\pm 7\ =\ 10$, or, -4. Ans.

Ex. 2.	Given.	$x^2-5x\quad=24$
	Complete the square.	$x^2-5x+6{\cdot}25=30{\cdot}25$
	Extract the square root.	$x\ -2{\cdot}5=\quad\pm\ 5{\cdot}5$
	Transpose.	$x\ =2{\cdot}5\pm 5{\cdot}5=\ 8$, or, -3. Ans.

Ex. 3.	Given (when reduced).	$x^2+\frac{4}{5}x=\frac{204}{5}$
	Complete the square.	$x^2-\frac{4x}{5}+\frac{4}{25}.=\frac{1024}{25}$
	Extract the square root.	$x+\frac{2}{5}=\pm\frac{32}{5}$
	Transpose.	$x=\pm\frac{32-2}{5}=6$, or $-6\frac{4}{5}$. Ans.*

Ex. 4.	Given.	$x^2-\frac{1}{2}x=14.$ (*By doubling and transposition.*)

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* Hutton's Answers in general to the Quadratic Equations are imperfect, in as much as he gives only ONE of the Roots of each Equation.

(Key to Vol. I. page 243.)

	Complete the square.	$x^2-\frac{1}{2}x+\frac{1}{16}=14\frac{1}{16}=14{\cdot}0625$
	Extract the square root.	$x-\frac{1}{4}=\pm3\frac{3}{4}$
	Transpose.	$x=\frac{1}{4}\pm3\frac{3}{4}=4$, or $-3\frac{1}{2}$. Ans.

Ex. 5.	Given.	$x^4-\frac{2}{3}x^2=\frac{40}{3}$
	Complete the square.	$x^4-\frac{2}{3}x^2+\frac{1}{9}=\frac{121}{9}$
	Extr. the square root.	$x^2-\frac{1}{3}=\pm3\frac{2}{3}$
	Transpose.	$x^2=\frac{1}{3}\pm3\frac{2}{3}=4$, or $-3\frac{1}{3}$ *an irrational Surd.*
	Extr. the square root.	$x=2$, or -2. Ans.

Ex. 6.	Given (× ing by 3).	$x-\frac{3}{2}\sqrt{x}=4\frac{1}{2}=\frac{9}{2}$
	Complete the square.	$x-\frac{3}{2}\sqrt{x}+\frac{9}{16}=5\frac{1}{16}$
	Extract the square root.	$\sqrt{x}-\frac{3}{4}=\pm2\frac{1}{4}$
	Transpose.	$\sqrt{x}=\frac{3}{4}\pm2\frac{1}{4}=3$, or $-1\frac{1}{2}$
	Square both sides.	$x=9$, or $2\frac{1}{4}$.* Ans.

* *When $x=2\frac{1}{4}$ its negative root must be taken to answer the conditions.*

Ex. 7.	Given.	$x^2+\frac{4}{3}x=1\frac{1}{2}$
	Complete the square.	$x^2+\frac{4}{3}x+\frac{4}{9}=\frac{70}{36}=1{\cdot}94444\dot{4}$
	Extract the square root.	$x+\frac{2}{3}=\pm1{\cdot}3944\dot{4}$.
	Transpose.	$x=\cdot727766$, or $-2{\cdot}0611\dot{1}$. Ans.

Ex. 8.	Given.	$x^6+4x^3=12$
	Complete the square.	$x^6+4x^3+4=16$
	Extract the square root.	$x^3+2=\pm4$
	Transpose.	$x^3=\pm4-2=2$, or -6.

(Key to Vol. I. page 243.)

Extr. the Cube root. | $x=\sqrt[3]{2}=1{\cdot}259921$, or $\sqrt[3]{-6}=-1.81712$.

Ex. 9. Given. | $x^2+4x=a^2+2$
Complete the square. | $x^2+4x+4=a^2+6$
Extract the square root. | $x+2=\pm\sqrt{a^2+6}$
Transpose. | $x=\pm\sqrt{a^2+6}-2$.

QUESTIONS FOR PRACTICE.

(Page 245.)

Quest. 1. Put | x for the number sought.
By the Question. | $x^2+x=42$
Complete the square. | $x^2+x+\frac{1}{4}=42\frac{1}{4}$
Extract the square root. | $x+\frac{1}{2}=\pm 6\frac{1}{2}$
Transpose. | $x=6$, or -7.

Quest. 2. Let x represent the greater number, and y the less.

By the Question $\left\{\begin{array}{l} y : x :: x : 12 \\ x^2+y^2 = 45 \end{array}\right\}$ or, $\left\{\begin{array}{l} x^2=12y \\ x^2=45-y^2 \end{array}\right\}$

Therefore | $y^2+12y=45$
Complete the sq^r^. | $y^2+12y+36=81$
Extract the sq^r^. root. | $y+6=\pm 9$
Transpose. | $y=3$, or, -15
But | $12y=x^2=36$, or, -180
Therefore | $x=6$, *or* $\sqrt{-180}$, *an irrational Surd.*

(Page 246.)

Quest. 3. Assume x as the greater, and y as the less number.

By the Question $\left\{\begin{array}{l} x=y+2 \\ x^3-y^3=98 \end{array}\right\}$ or, $\left\{\begin{array}{l} x^3=y^3+6y^2+12y+8 \\ x^3=y^3+98 \end{array}\right\}$

Therefore | $6y^2+12y=90$
Divide by 6. | $y^2+2y=15$
Complete the square. | $y^2+2y+1=16$
Extract the square root. | $y+1=\pm 4$
Transpose. | $y=3$, or, -5
But | $x=y+2=5$, or -3. } Ans.

(Key to Vol. I. page 246.)

Quest. 4. Put x and y for the two numbers sought.

$$\text{By the Quest.}\left\{\begin{array}{l} x+y=6 \\ x^3+y^3=72 \end{array}\right\}\text{ or, }\left\{\begin{array}{l} x^3=216-108y+18y^2-y^3 \\ x^3=72 \qquad\qquad\qquad\qquad\quad -y^3 \end{array}\right\}$$

$$\begin{array}{r|l}
\text{Therefore} & 18y^2-108y=-144 \\
\text{Divide by 18.} & y^2-6y=-8 \\
\text{Complete the square.} & y^2-6y+9=1 \\
\text{Extract the square root.} & y-3=\pm1 \\
\text{Transpose.} & y=4,\text{ or } 2. \\
\text{But} & x=6-y=2,\text{ or } 4.
\end{array}$$

That is, no numbers except 4 *and* 2 *will answer the conditions of the Question.*

Quest. 5. Let x and y represent the two numbers proposed.

$$\text{By the Question}\left\{\begin{array}{l} xy=20 \\ x^3-y^3=61 \end{array}\right\}\text{ or, }\left\{\begin{array}{l} x=\frac{20}{y},\text{ and } x^3=\frac{8000}{y^3} \\ \text{---------}\ x^3=y^3+61. \end{array}\right.$$

$$\begin{array}{r|l}
\text{Therefore} & y^3+61=\frac{8000}{y^3} \\
\text{Reduce.} & y^6+61y^3=8000
\end{array}$$

$$\begin{array}{r|l}
\text{Complete the square.} & y^6+61y^3+930\frac{1}{4}=8930{\cdot}25 \\
\text{Extract the square root.} & y^3+30\frac{1}{2}=\pm94{\cdot}5 \\
\text{Transpose.} & y^3=64.\text{ or } -125 \\
\text{Extract the cube root.} & y=4.\text{ or } -5 \\
\text{But} & xy=20=4x,\text{ or, } -5x \\
\text{Consequently} & x=5.\text{ or } -4.
\end{array}$$

Quest. 6.

$$\text{Given}\left\{\begin{array}{l} x+y=11 \\ x^2y^2=784 \end{array}\right\}\text{ or, }\left\{\begin{array}{l} x=11-y \\ x=\frac{28}{y} \end{array}\right\}$$

$$\begin{array}{r|l}
\text{Therefore} & 11-y=\frac{28}{y} \\
\text{Reduce.} & y^2-11y=-28 \\
\text{Complete the square.} & y^2-11y+30{\cdot}25=2{\cdot}25 \\
\text{Extract the square root.} & y-5{\cdot}5=\pm1{\cdot}5 \\
\text{Transpose.} & y=7,\text{ or } 4. \\
\text{But} & x=\frac{28}{y}=4.\text{ or } 7.
\end{array}$$

(Key to Vol. I. page 246.)

Quest. 7.

$$\text{Given}\left\{\begin{matrix} x+y=5 \\ \frac{x}{y}+\frac{y}{x}=4\frac{1}{4} \end{matrix}\right\}\text{or,}\left\{\begin{matrix} x=5-y \\ \frac{x^2+y^2}{xy}=4\frac{1}{4} \end{matrix}\right\}$$

Substitute $5-y$ for x.	$\frac{25-10y+2y^2}{5y-y^2}=4\frac{1}{4}$
Multiply by $5y-y^2$.	$2y^2-10y+25=21\frac{1}{4}y-4\frac{1}{4}y^2$
Transpose and collect.	$6\frac{1}{4}y^2-31\frac{1}{4}y=-25$
Divide by $6\frac{1}{4}$.	$y^2-5y=-4$
Complete the sq^r and extract.	$y-2\frac{1}{2}=\pm1\frac{1}{2}$
Transpose.	$y=4,$ or $1.$ } Ans.
Consequently	$x=1,$ or $4.$ } Ans.

Quest. 8.

$$\text{Given}\left\{\begin{matrix} x+y=12 \\ xy=8x-8y \end{matrix}\right\}\text{or,}\left\{\begin{matrix} x=12-y \\ \text{-------} \end{matrix}\right\}$$

Substitute $12-y$ for x in the 2d equat.	$12y-y^2=96-16y$
Change the signs, and reduce.	$y^2-28y=-96$
Complete the square.	$y^2-28y+196=100$
Extract the square root.	$y-14=\pm10$
Transpose.	$y=24$. (*not applicable*), or 4.
Now	$x=12-y=12-4=8$.

Quest. 9. Let x represent the greater part, y the less.

$$\text{By the Quest.}\left\{\begin{matrix} x+y=10 \\ 16y^2-112=4x^2 \end{matrix}\right\}\text{hence}\left\{\begin{matrix} 4x^2=400-80y+4y^2 \\ 4x^2=16y^2-112 \end{matrix}\right\}$$

Therefore, and by reduction,	$y^2+\frac{20}{3}y=42\frac{2}{3}$
Complete the square, extract, and transpose.	$y=4$. or $-10\frac{2}{3}$ *inapplicable.*
Consequently	$x=6$.

Quest. 10. Put x and y for the numbers required, x being the greater.

$$\text{By the Quest.}\left\{\begin{matrix} x^2+xy=104 \\ x^2+y^2=89 \end{matrix}\right\}\text{or,}\left\{\begin{matrix} x^2=104-xy \\ x^2=89-y^2\text{, and } x=\sqrt{89-y^2} \end{matrix}\right.$$

(Key to Vol. I. page 246.)

Therefore	$y^2+15=xy=y\sqrt{89-y^2}$
Square both sides.	$y^4+30y^2+225=89y^2-y^4$
Transpose.	$2y^4-59y^2=-225$
Divide by 2, and complete.	$y^4-\frac{59}{2}y^2+\frac{3481}{16}=\frac{1681}{16}$
Extract the square root.	$y^2-\frac{59}{4}=\pm\frac{41}{4}$
Transpose.	$y^2=\frac{59}{4}\pm\frac{41}{4}=25$, or $4\frac{1}{2}$.

Extract the square root.	$y=5.$ or $2\cdot12131$ &c.
But	$x=\sqrt{89-y^2}=8.$ or $9\cdot19238$ &c.

Quest. 11. Let x be the 1st digit, y the 2d, and z the number sought.

$$\text{By the Question}\left\{\begin{array}{l}10x+y=z\\ \dfrac{10x+y}{xy}=5\frac{1}{3}\end{array}\right\}\text{also}\left\{10x+y-9=10y+x\right.$$

Collect the last equat. and divide by 9.	$x=y+1$
Substitute $y+1$ for x in the 2d equat.	$\frac{11y+10}{y^2+y}=5\frac{1}{3}$
Multiply by y^2+y.	$11y+10=5\frac{1}{3}y^2+5\frac{1}{3}y$
Reduce, and divide by $5\frac{1}{3}$.	$y^2-\frac{17}{16}y=\frac{30}{16}$
Complete the square, extract the sq^r. root, and transpose.	$y=2$
But	$x=y+1=3$
And	$z=10x+y=30+2=32.$

Quest. 12. Let the three parts be severally represented by x, y, and z.

$$\text{By the Question}\left\{\begin{array}{l}x+y+z=20\\ xyz=270\\ x-(y+2)=y-z\end{array}\right\}\text{or,}\left\{\begin{array}{l}x=20-y-z\\ x=\dfrac{270}{yz}\\ x=2y-z+2.\end{array}\right\}$$

(Key to Vol. I. page 246.)

Comparing the 1st and 3d values of x.	$18=3y$, or $y=6$
Substitute 6 for y in the 2d equation.	$x=\frac{270}{6z}$
Compare this with the 1st equat. substituting 6 for y.	$\frac{270}{6z}=14-z$
Reduce, divide by 6, and change all the signs.	$z^2-14z=-45$
Complete the square.	$z^2-14z+49=4$
Extract the square root, and transpose.	$z=9$, or 5.

$$\text{But } x=2y-z+2=12+2-\begin{Bmatrix}9\\5\end{Bmatrix}\begin{matrix}=5\\=9\end{matrix}$$

The difference of x and y, however, being limited by the question to be two less than the difference of y and z, it follows that $x=5$, $y=6$, and $z=9$.

Quest. 13. Assume x, y, and z. Then

Since x, y, and z are in arithmetical progression $x+z=2y$.

$$\text{By the Question, also, }\begin{Bmatrix}x^2+y^2+z^2=56\\3x+2y+3z=32\end{Bmatrix}$$

In the 1st equat. transpose z, and square.	$x^2=4y^2-4yz+z^2$
In the 2d equat. transp. y^2+z^2.	$x^2=56-y^2-z^2$
In the 3d equation transpose $2y-3z$, and divide by 3.	$x=\frac{32-2y-3z}{3}$
Comparing this value of x with the 1st equat.	$8y=32$, or $y=4$.
Compare the two values of x^2, substituting 4 for y.	$2z^2-16z=-24$.
Divide by 2, and complete the square.	$z^2-8z+16=4$
Extract the square root, and transpose.	$z=6$. or 2.
But	$x=\frac{32-2y}{3}-z=8-z=2$. or 6.

Therefore 2, 4, 6, *or* 6, 4, 2, *are the series required.*

Quest. 14.

$$\text{Given}\begin{Bmatrix}x+y+z=13\\x^2-y^2=y^2-z^2\\x^2+y^2+z^2=75\end{Bmatrix}\text{or, }\begin{Bmatrix}x^2=(13-y-z)^2\\x^2=2y^2-z^2\\x^2=75-y^2-z^2\end{Bmatrix}.$$

(Key to Vol. I. page 246.)

Compare the 2d and 3d equat^s.	$3y^2=75$, or $y=5$
Compare the 1st and 2d equat^s. substituting 5 for y.	$2z^2-16z=-14$.
Divide by 2, and complete the square.	$z^2-8z+16=9$
Extract the sq^r root, and transp.	$z=7$ or 1
But	$x=13-y-z=8-z=1$ or 7.
Therefore	1, 5, and 7 are *the parts sought.*

Quest. 15. Put x, y, and z for the three numbers proposed.

By the Question $\left\{\begin{array}{l} x+y+z=12 \\ x-y=y-z \\ x^4+y^4+z^4=962 \end{array}\right\}$ or, $\left\{\begin{array}{l} x=12-y-z \\ x=2y-z \\ \text{- - - - - - - - -} \end{array}\right\}$

Compare these two values of x.	$12=3y$, or $y=4$
Substitute 4 for y in the 2d and 3d equations	$x+z=8$, and $x^4+z^4=706$
Put $a=\dfrac{x+z}{2}$ and $n=\dfrac{x-z}{2}$.	$x=a+n$, and $z=a-n$.
Therefore	$x^4=\overline{a+n}^4=a^4+4a^3n+6a^2n^2+4an^3+n^4$ $z^4=\overline{a-n}^4=a^4-4a^3n+6a^2n^2-4an^3+n^4$
Add, and divide by 2.	$353=a^4+6a^2n^2+n^4$
Substitute 4 (its known value) for a.	$353=256+96n^2+n^4$
Transpose, and complete the square.	$n^4+96n^2+2304=2401$
Extract the square root, and transpose.	$n^2=\pm 49-48=1$, or -97 *an irrat. Surd.*
Extract the square root.	$n=\pm 1$.

But $x=a+n=4\pm 1=5$ or 3, and $z=a-1=4\mp 1=3$, or 5.

Consequently 3, 4, and 5, are the numbers sought.

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Quest. 16. For the proposed three numbers, substitute x, y, and z.

By the Question $\left\{\begin{array}{l} 2y=x+z \\ z^2+xy=28 \\ x^2+yz=44 \end{array}\right\}$ or, $\left\{\begin{array}{l} y=\frac{1}{2}(x+z) \\ \text{- - - - -} \\ \text{- - - - -} \end{array}\right\}$

(Key to Vol. I. page 247.)

Sub. $\frac{x+z}{2}$ for y in the 2d and 3d equat^{s}.	$z^2+\frac{x^2+xz}{2}=28$, or, $2z^2+x^2+xz=56$
- - - - - - - - - - - - -	$x^2+\frac{z^2+xz}{2}=44$, or, $2x^2+z^2+xz=88$
By subtraction.	$x^2=z^2+32$
Put $r+s=x$.	$x^2=r^2+2rs+s^2$
and $r-s=z$.	$z^2+32=r^2-2rs+s^2+32$
Therefore (Eucl. i. Ax. 1.)	$4rs=32$, and $r=\frac{8}{s}$.

But

$$x=r+s=\frac{8}{s}+s=\frac{8+s^2}{s}$$

And

$$z=r-s=\frac{8}{s}-s=\frac{8-s^2}{s}.$$

Also

$$x^2+\frac{xz+z^2}{2}=44=\frac{64+16s^2+s^4}{s^2}+\frac{64-s^4+16s^2+64+s^4}{2s^2}$$

Reduce.

$$44=\frac{s^4+8s^2+128}{s^2}, \text{ or } 44s^2=s^4+8s^2+128.$$

By further reduction.

$$s^4-36s^2=-128.$$

Complete the square.

$$s^4-36s^2+324=196.$$

Extract the square root and transpose.

$$s^2=32, \text{ or } 4.$$

Extract the square root.

$$s=5{\cdot}6568542 \text{ \&c. or } 2.$$

Wherefore,

$$r=\frac{8}{s}=\frac{8}{\sqrt{32}}, \text{ or } \frac{8}{2}=4.$$

And

$$x=r+s=\frac{8}{\sqrt{32}}+\sqrt{32}, \text{ or } 4+2=6. \quad \text{Likewise}$$

$$z=r-s=\frac{8}{\sqrt{32}}-\sqrt{32}, \text{ or } 4-2=2.$$

Rejecting the Surd values of x and z as irrelevant to the question, $y=\frac{x+z}{2}=\frac{6+2}{2}=4$. *Consequently* 2, 4, 6 *are the numbers required.*

Quest. 17. Given $\left\{\begin{array}{lcr} \text{A}+\text{B}+\text{C} & = & £1444 \\ \text{B}+\sqrt{\text{A}} & = & 920 \\ \text{B}+\sqrt{\text{C}} & = & 912 \end{array}\right\}$

Put $\left\{\begin{array}{l} x^2+2xy+y^2=\text{A} \\ x^2-2xy+y^2=\text{C} \end{array}\right\}$

Then by comparison and substitution,

$$£1444-x^2-2xy-y^2-x^2+2xy-y^2+x+y=£920.$$
$$£1444-x^2-2xy-y^2-x^2+2xy-y^2+x-y=912.$$

By subtraction - - - - - - - - - - - - - - - $2y=8$. And $£2888-4x^2-4y^2+2x=1832$. by addition.

Transpose, and substitute £8 for $2y$.

$£992=4x^2-2x$, hence $£248=x^2-\frac{1}{2}x$.

Complete the sq^{r}, extract the square root, and transpose.

$x=£16$.

But $y=£4$. Therefore

$x+y=£20=\sqrt{\text{A}}$.

And

$x-y=£12=\sqrt{\text{C}}$.

Consequently $\text{A}=£400$, $\text{B}=£900$, *and* $\text{C}=£144$.

Quest. 18. Let x, y, and z, represent the three numbers sought.

By the Question. $\left\{\begin{array}{l} x+z=2y \\ x^2+y^2+z^2=93 \\ 3x+4y+5z=66 \end{array}\right\}$ or $\left\{\begin{array}{l} x=2y-z \\ \text{- - - - - - - - -} \\ x=\frac{66-4y-5z}{3} \end{array}\right\}$

Compare these two values of x.

$10y+2z=66$, or $y=\frac{66-2z}{10}$.

Substitute in the 1st equation $\frac{66-2z}{10}$ for y.

$x=\frac{66-2z}{5}-z=\frac{66-7z}{5}$.

x and y being now known in terms of z, it is

$$x^2+y^2+z^2=\frac{4356-924z+49z^2}{25}+\frac{4356-264z+4z^2}{100}+z^2=93.$$

Reduce, complete the square, extract the square root, and transpose, $z=8$.

But

$$y=\frac{66-2z}{10}=\frac{66-16}{10}=\frac{50}{10}=5.$$

And

$$x=2y-z=10-8=2.$$

Therefore

2, 5, *and* 8, *are the numbers required.*

Quest. 19. For the two numbers proposed, put x and y.

By the Question. $\left\{\begin{array}{l} xy+x+y=47 \\ x^2+y^2-x-y=62 \end{array}\right\}$

Assume $m=\frac{1}{2}(x+y)$ and $n=\frac{1}{2}(x-y)$.

By substitution.	$xy+x+y=m^2-n^2+2m=47$
And	$x^2+y^2-(x+y)=2m^2+2n^2-2m=62$
Subtr. half the last equat. from the equat. above it.	$3m-2n^2=16$
Add the last subtrahend to its minuend.	$2m^2+m=78$
Divide by 2, and complete the square.	$m^2+\frac{1}{2}m+\frac{1}{16}=39{\cdot}0625$
Extract the square root and transpose.	$m=6$
But	$3m-2n^2(=18-2n^2)=16$
Therefore	$2n^2=2$, or $n=1$.

Now $m+n=x=7$, and $m-n=y=5$. *Which were to be determined.*

(Key to Vol. I. page 253.)

CUBIC AND HIGHER EQUATIONS.

Ex. 5. Assume 4 and 5 as values of x.

$$x\text{ being }4\left\{\begin{array}{rcccr} 20 & = & 5x & = & 25\\ 160 & = & 10x^2 & = & 250\\ 64 & = & x^3 & = & 125\\ \cline{1-1}\cline{5-5} 244 & & \text{Sum} & & 400\\ 260 & & \textit{should be} & & 260\\ \cline{1-1}\cline{5-5} \end{array}\right\}x\text{ being }5.$$

Too little 16 Errors 140 *Too much.*

The sum of the errors is 156. And

As 156 : 1. : : 16 : ·1. Hence $x=4{\cdot}1$ nearly.

Proceeding now with 4·1 and 4·2, x is found 4·1179 &c. Ans.

Ex. 6. Assume 3·5 and 3·9 as values of x.

$$x\text{ being }3{\cdot}5\left\{\begin{array}{rcccr} 42{\cdot}875 & = & x^3 & = & 59{\cdot}319\\ -7 & = & -2x & = & -7{\cdot}8\\ \cline{1-1}\cline{5-5} 35{\cdot}875 & & \text{Sum} & & 51{\cdot}519 \end{array}\right\}x\text{ being }3{\cdot}9$$

$$\begin{array}{rcr} 50 & \textit{should be} & 50{\cdot}\\ \hline \end{array}$$

Too little 14·125 Errors 1·519 *Too much.*

The sum of the errors is 15·644. And

As 15·644 : ·4 : : 1·519 : ·038839 *the excess of* 3·9.

But $3{\cdot}9-{\cdot}038839=3{\cdot}8611=x$ *nearly.*

Proceeding, therefore, with 3·861 and 3·862, x is found *more nearly* 3·8648. Ans.

(Page 254.)

Ex. 7. By a few trials x is discovered to be between 5·1 and 5·2. Wherefore assuming these,

$$x\text{ being }5{\cdot}1\left\{\begin{array}{rcccr} -117{\cdot}3 & = & -23x & = & -119{\cdot}6\\ 52{\cdot}02 & = & 2x^2 & = & 54{\cdot}08\\ 132{\cdot}625 & = & x^3 & = & 140{\cdot}608\\ \cline{1-1}\cline{5-5} \end{array}\right\}x\text{ being }5{\cdot}2.$$

$$\begin{array}{rcr} 67{\cdot}345 & \text{Sum} & 75{\cdot}088\\ 70{\cdot} & \textit{should be} & 70{\cdot}\\ \hline \end{array}$$

Too little 2·655 Errors 5·088 *Too much.*

(Key to Vol. I. page 254.)

The sum of the errors is 7·743. And
As 7·743 : ·1 : : 2·655 : ·03428 *the deficiency of* 5·1.
But $5{\cdot}1+{\cdot}03428=5{\cdot}13428=x$ *nearly.*
By another approximation x is found (*more nearly*) 5·13457.

Ex. 8. Assuming 14·9 and 15 as values of x.

$$x \text{ being } 14{\cdot}9\left\{\begin{array}{rcrcr} 3307{\cdot}949 & = & x^3 & = & 3375\\ -3774{\cdot}17 & = & -17x^2 & = & -3825\\ 804{\cdot}6 & = & 54x & = & 810\\ \cline{1-1}\cline{5-5} 338{\cdot}379 & & \text{Sum} & & 360\end{array}\right\}x \text{ being } 15.$$

$$\begin{array}{rrcrl} & 350{\cdot} & \textit{should be} & 350 & \\ \cline{2-2}\cline{4-4} \textit{Too little} & 11{\cdot}621 & \text{Errors} & 10 & \textit{Too much.}\end{array}$$

The sum of the errors is 21·621. And
As 21·621 : ·1 : : 10 : ·04625 *the excess of* 15.
But $15-{\cdot}04625=14{\cdot}9537=x$ *nearly.*
By another approximation x is found (*more nearly*) 14·95407.

Ex. 9. A few trials determine x between 10·2 and 10·3,
Let these, therefore, be assumed as x. It is

$$x \text{ being } 10{\cdot}2\left\{\begin{array}{rcrcr} 10824{\cdot}3226 & = & x^4 & = & 11255{\cdot}0881\\ -312{\cdot}12 & = & -3x^2 & = & -318{\cdot}27\\ -765 & = & -75x & = & -772{\cdot}5\\ \cline{1-1}\cline{5-5} \end{array}\right\}x \text{ being } 10{\cdot}3.$$

$$\begin{array}{rrcrl} & 9747{\cdot}2026 & \text{Sum} & 10164{\cdot}3181 & \\ & 10000{\cdot} & \textit{should be} & 10000{\cdot} & \\ \cline{2-2}\cline{4-4} \textit{Too little} & 252{\cdot}7974 & \text{Errors} & 164{\cdot}3181 & \textit{Too much.}\end{array}$$

The sum of the errors is 417·1155. And
As 417·1155 : ·1 : : 164·3181 : ·03939 *the excess of* 10·3.
But $10{\cdot}3-{\cdot}03939=10{\cdot}2606=x$ *nearly.*
Yet another approximation determines x (*more nearly*) 10·2609.

Ex. 10. After a few trials assuming 1·284 and 1·285 as values of x, it is

$$\begin{array}{c}x \text{ being}\\ 1{\cdot}284\end{array}\left\{\begin{array}{rcrcr} 5{\cdot}436153 & = & 2x^4 & = & 5{\cdot}453085\\ -33{\cdot}869988 & = & -16x^3 & = & -33{\cdot}949186\\ 65{\cdot}94624 & = & 40x^2 & = & 66{\cdot}049\\ -38{\cdot}52 & = & -30x & = & -38{\cdot}55\\ \cline{1-1}\cline{5-5} \end{array}\right\}\begin{array}{c}x \text{ being}\\ 1{\cdot}285.\end{array}$$

$$\begin{array}{rrcrl} & -\ 1{\cdot}007595 & \text{Sum} & -\ 0{\cdot}997101 & \\ & -\ 1{\cdot} & \textit{should be} & -\ 1{\cdot} & \\ \cline{2-2}\cline{4-4} \textit{Too much} & {\cdot}007595 & \text{Errors} & {\cdot}002899 & \textit{Too little.}\end{array}$$

(Key to Vol. I. page 254.)

The sum of the errors is ·010494. And
As ·010494 : ·001 : : ·002899 : ·00027 *the excess* of* 1·285.
But $1{\cdot}285-{\cdot}00027=1{\cdot}28473=x$ *very nearly.*

* *The excess because the negative quantities exceed the positive.*

Ex. 11. After several trials it is discovered that x is between 8·41 and 8·42.

Assuming these, therefore, as values of x, it will be

$$x\text{ being }8{\cdot}41\left\{\begin{array}{rcl} 42070{\cdot}72333 & =x^5= & 42321{\cdot}442067\\ 10004{\cdot}928259 & =2x^4= & 10052{\cdot}598065\\ 1784{\cdot}469963 & =3x^3= & 1790{\cdot}843064\\ 282{\cdot}9124 & =4x^2= & 283{\cdot}5856\\ 42{\cdot}05 & =5x= & 42{\cdot}1 \end{array}\right\}x\text{ being }8{\cdot}42$$

$$\begin{array}{rcl} \hline 54185{\cdot}082952 & \text{Sum} & 54490{\cdot}568796\\ 54321{\cdot} & \textit{should be} & 54321{\cdot}\\ \hline \textit{Too little}\ \ 135{\cdot}917048 & \text{Errors} & 169{\cdot}568796\ \ \textit{Too much.} \end{array}$$

The sum of the errors is 305·485844. And
As 305·485844 : ·01 : : 135·917048 : ·00445 *the deficiency of* 8·41.
But $8{\cdot}41+{\cdot}00445=8{\cdot}41445=x$ *nearly.*

Ex. 12. Assume 8·5 and 8·6 as values of x.

$$\begin{array}{l} x\text{ being}\\ 8{\cdot}5 \end{array}\left\{\begin{array}{r} \text{Log. of }8{\cdot}5\text{ is }{\cdot}9294189\\ \text{Multiply by}\qquad 8{\cdot}5\\ \hline \text{Product }7{\cdot}9000606 \end{array}\right.\qquad\left.\begin{array}{l} {\cdot}9344985\text{ Log. of }8{\cdot}6\\ \text{Mult. by }8{\cdot}6\\ \hline 8{\cdot}0366871\text{ Product} \end{array}\right\}\begin{array}{l} x\text{ being}\\ 8{\cdot}6 \end{array}$$

$$\begin{array}{rcl} \text{Log. of }123456789\text{ is }8{\cdot}0915150 & & 8{\cdot}0915150\text{ Log. of }123456789\\ \hline \textit{Too little}\ \ {\cdot}1914543 & \text{Errors} & {\cdot}0548279\ \ \textit{Too little.} \end{array}$$

The difference of the errors is ·1366264. And
As ·1366264 : ·1 : : ·0548279 : ·04 *the deficiency of* 8·6.
But $8{\cdot}6+{\cdot}04=8{\cdot}64=x$ *nearly.*

Proceeding therefore with 8·64 and 8·641 as values of x, it is,

$$\begin{array}{l} x\text{ being}\\ 8{\cdot}64 \end{array}\left\{\begin{array}{r} \text{Log of }8{\cdot}64\text{ is }{\cdot}9365137\\ \text{Mult. by}\qquad 8{\cdot}64\\ \hline \text{Product }8{\cdot}0914783 \end{array}\right.\qquad\left.\begin{array}{l} {\cdot}936564\text{ Log. of }8{\cdot}641\\ \text{Mult. by }8{\cdot}641\\ \hline 8{\cdot}092849\text{ Product} \end{array}\right\}\begin{array}{l} x\text{ being}\\ 8{\cdot}641 \end{array}$$

$$\begin{array}{rcl} \text{Log. of }123456789\text{ is }8{\cdot}0915150 & & 8{\cdot}091515\text{ Log. of }123456789\\ \hline \textit{Too little}\ \ {\cdot}0000367 & \text{Errors} & {\cdot}001334\ \ \textit{Too much.} \end{array}$$

The sum of the errors is ·001371. Also

As ·001371 : ·001 : : ·0000367 : ·0000268 *the deficiency* 8·64.

But $8\cdot64+\cdot0000268=8\cdot6400268=x$. Ans.

Ex. 13. After a few trials assuming 1·84 and 1·85, as values of x,

$$x\text{ being }1\cdot84\left\{\begin{array}{l}22\cdot92457472\\ -43\cdot606528\\ \ \ 37\cdot2416\\ -\ \ 5\cdot52\end{array}\right. \begin{array}{l}=\ \ 2x^4=\\ =-7x^3=\\ =\ \ 11x^2=\\ =-3x\ =\end{array} \left.\begin{array}{l}\ \ 23\cdot4270125\\ -44\cdot321375\\ \ \ 37\cdot6475\\ -\ \ 5\cdot55\end{array}\right\}x\text{ being }1\cdot85$$

	11·03964672	Sum	11·2031375	
	11·	*should be*	11·	
Too much	·03964672	Errors	·2031375	*Too much.*

The difference of the errors is ·1634907. Now

As ·1634907 : ·01 : : ·03964672 : ·00242 *the excess of* 1·84.

But $1\cdot84-\cdot00242=1\cdot83758=x$ *nearly*.

Ex. 14. Let 18 and 19 be severally assumed and substituted for x.

x being 18. x being 19.

$$\left\{\begin{array}{l}\sqrt[5]{964\cdot5147186}^{\,3}=\ (3x^2-2\sqrt{x}+1)^{\frac{3}{5}}\ =\sqrt[5]{1075\cdot2822022}^{\,3}\\ -\sqrt[9]{31\cdot2577917}^{\,5}=-(x^2-4x\sqrt{x}+3\sqrt{x})^{\frac{5}{9}}=-\sqrt[9]{42\cdot8003803}^{\,5}\end{array}\right\}$$

But,

Log. of 964·5147186 is 2·9843088. which multiplied by 3, divided by 5, becomes

(*Log. of* 61·7426571. *or*) 1·7905853 the log. of the 5th root of the cube of 964·5147186. And

Log. of 31·2577917 is 1·4949583. which multiplied by 5, divided by 9, becomes

(*Log. of* 6·7691234. *or*) 0·8305324 the log. of the 9th root of the 5th power of 31·2577917. Also

Log. of 1075·2822022 is 3·0315234 which multiplied by 3, divided by 5, becomes

(*Log. of* 65·904338. *or*) 1·8189140 the log. of the 5th root of the cube 1075·2822022. Again

(Key to Vol. I. page 254.)

Log. of 42·8003803 is 1·6314476 which multiplied by 5, divided by 9, becomes
(*Log. of* 8·060459. *or*) 0·9063598 the log. of the 9th root of the 5th power of 42·8003803.

Substituting these four natural numbers found, it is,

61·7426571·		65·904338
− 6·7691234		− 8·060459
54·9735337	Sum	57·843879
56·	*should be*	56·
Too little 1·0264663	Errors	1·843879 *Too much.*

The sum of the errors is 2·8703453. And

As 2·8703453 : 1 : : 1·0264663 : ·357 *the deficiency of* 18.

But 18+·357=18·357=x *nearly.*

Proceeding with 18·357 and 18·358, x is found (*more nearly*) 18·36087 &c. Ans.

CARDAN'S RULE FOR CUBIC EQUATIONS.

(Page 256.)

Ex. 2. Substituting $z+3$ for x

$$\begin{aligned} x^3 &= z^3+9z^2+27z+27 \\ -9x^2 &= \quad -9z^2-54z-81 \\ +28x &= \qquad\quad +28z+84 \end{aligned}$$

$$\text{Therefore } z^3 \;\; * \;\; +\; z \; +30=30$$

That is, $z^3+z=0$, or $z=0$, consequently $x=3$ in one of the roots of the given equation.

Then,

$x-3)\,x^3-9x^2+28x-30\,(=x^2-6x+10$, *the equation depressed one degree.*

By transposition.	$x^2-6x= \quad -10$
Complete the square.	$x^2-6x+\;9=-1$
Extract the square root.	$x\;-3\;=\pm\sqrt{-1}$
Transpose.	$x\;=3\;\pm\sqrt{-1}$

Therefore 3, $3+\sqrt{-1}$, *and* $3-\sqrt{-1}$ *are the three values of* x.

(Key to Vol. I. page 256.)

Ex 3. Substitute $z+\frac{7}{3}$ for x;

$$\begin{aligned} x^3 &= z^3+7z^2+\tfrac{147}{9}z+\tfrac{343}{27} \\ -\ 7x^2 &= \quad -7z^2-\tfrac{294}{9}z-\tfrac{1029}{27} \\ +14x &= \qquad\qquad +\tfrac{126}{9}z+\tfrac{882}{27} \end{aligned}$$

Therefore $z^3 \ * \ -\frac{7}{3}z+\frac{196}{27}=20$, or, $z^3-\frac{7}{3}z=12\frac{20}{27}$.

Here $a=-\frac{7}{3}$, $b=12\frac{20}{27}$, $c=-\frac{7}{9}$, and $d=6\frac{10}{27}$. Whence

$$\sqrt[3]{d+\sqrt{d^2+c^3}}+\sqrt[3]{d-\sqrt{d^2+c^3}}=\sqrt[3]{\frac{172}{27}+\sqrt{\frac{29584-343}{729}}}+$$

$$\sqrt[3]{\frac{172}{27}-\sqrt{\frac{29584-343}{729}}}=\sqrt[3]{\frac{172+171}{27}}+\sqrt[3]{\frac{172-171}{27}}=\sqrt[3]{\frac{343}{27}}+$$

$\sqrt[3]{\frac{1}{27}}=\frac{7}{3}+\frac{1}{3}=\frac{8}{3}=2\frac{2}{3}=z$. But $z+\frac{7}{3}=x=2\frac{2}{3}+2\frac{1}{3}=5$ the value of the unknown quantity in one of the roots of the given equation.

Then,

$x-5)\ x^3-7x^2+14x-20\ (=x^2-2x+4$ *the equation depressed one degree.*

By transposition.	$x^2-2x \quad =-4$
Complete the square.	$x^2-2x+1=-3$
Extract the square root.	$x\ -1=\pm\sqrt{-3}$
Transpose.	$x\ =1\ \ \pm\sqrt{-3}$.

Therefore 5, $1+\sqrt{-3}$, *and* $1-\sqrt{-3}$ *are the three values of* x.

APPLICATION OF ALGEBRA TO GEOMETRY.

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PROBLEM VI.

Put x for the hypothenuse,
b for the base (3) of the triangle; and
d for the difference (1) of the hypothenuse and perpendicular.

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Then (because Eucl. i. 18. the hypothenuse is greater than the perpendicular) $x-d=$the perpendicular.

But (Eucl. i. 47.) $x^2=(x-d)^2+b^2=x^2-2dx+d^2+b^2$, that is, $2dx=d^2+b^2$ and $x=\frac{d^2+b^2}{2d}=$ *(substituting known values)* $\frac{1^2+3^2}{2\times 1}=\frac{10}{2}=5$.

Wherefore the triangle is determined.

PROBLEM VII.

Put x for the less side about the right angle,
d for the difference (1) of the base and perpendicular,
also a for the hypothenuse (5).

Then (Eucl. i. 47.) $a^2=x^2+(x+d)^2=x^2+x^2+2dx+d^2=2x^2+2dx+d^2$, that is, $x^2+dx=\frac{a^2-d^2}{2}$. Completing the square, extracting the square root, and transposing, $x=-\frac{1}{2}d\pm\sqrt{\frac{a^2-d^2}{2}+\frac{1}{4}d^2}$. $=$*(substituting known values)* $-\frac{1}{2}\pm\sqrt{\frac{25-1}{2}+\frac{1}{4}}=-\frac{1}{2}\pm 3\frac{1}{2}=3$ or -4, of which values the last (being irrelevant) is to be rejected.

Wherefore, x being known, the triangle is determined.

PROBLEM VIII.

If a be put for the perpendicular, and
b for the base of the triangle;
c for the area of the rectangle; and
x for the length of the rectangle parallel to the base of the triangle, it will be

As $b:a::x:\frac{ax}{b}$ the difference of the altitudes of the rectangle and triangle.

Therefore the altitude of the rectangle is $a-\frac{a}{b}x$.

But $c=x\ (a-\frac{a}{b}x)=ax-\frac{a}{b}x^2$, that is, $x=\frac{1}{2}b\pm\sqrt{\frac{b^2}{4}-\frac{b}{a}c}$ *the*

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length of the rectangle; consequently $\dfrac{c}{\frac{1}{2}b\pm\sqrt{\frac{b^2}{4}-\frac{b}{a}c}}$ = *the breadth; which were to be determined.*

PROBLEM IX.

Put a and b for the two segments of the base,
x for the side adjacent to a,
y for the other side of the triangle ; and
$m : n$, the ratio x has to y.

Then, (since $m : n :: x : y$) it follows that $y=\frac{n}{m}x$.

But $x^2-a^2=y^2-b^2=\dfrac{n^2x^2}{m^2}-b^2$.

That is, $m^2x^2-n^2x^2=m^2a^2-m^2b^2$.

$$\left.\begin{aligned}&\text{Hence } x=\sqrt{\frac{m^2a^2-m^2b^2}{m^2-n^2}}\\ &\text{And } \quad y=\frac{n}{m}\sqrt{\frac{m^2a^2-m^2b^2}{m^2-n^2}}\end{aligned}\right\}\textit{which were to be determined.}$$

PROBLEM X.

Put b for half the base,
c for the line from the vertex,
a for the sum of the two sides ; and
x for one of the sides.

Then $a-x$ is the other side of the triangle.
But (Theor. 38) $2b^2+2c^2=a^2-2ax+2x^2$. Wherefore $x=\frac{1}{2}a\pm\sqrt{b^2+c^2-\frac{1}{4}a^2}$, and $a-x=\frac{1}{2}a\mp\sqrt{b^2+c^2-\frac{1}{4}a^2}$.
Consequently the triangle is determined.

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PROBLEM XI.

Let a represent one of the sides of the triangle,
b the other side,
c the line bisecting the vertical angle, and
x the base;

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it will be, As $(a+b) : x :: a : \frac{ax}{a+b}$ the segment of the base, adjacent to a.

And, As $(a+b) : x :: b : \frac{bx}{a+b}$ the seg. of the base, adjacent to b.

Wherefore *(Because in any triangle having the vertical angle bisected, the rectangle of the two sides less the rectangle of the segments of the base is equal to the square of the line bisecting the vertical angle)* [Theor. lxiv.] $ab - (\frac{ax \times bx}{a+b}) = c^2$.

Hence $x^2 = \frac{a^3b + 2a^2b^2 + ab^3 - a^2c^2 - 2a^2cb - b^2c^2}{ab}$; which equation resolved, $x = (a+b)\sqrt{\frac{ab-c^2}{ab}}$ = *the base of the triangle* Q. E. I.

PROBLEM XII.

If a be put for the line joining the acute angle at the base and the middle of the perpendicular,
b for the line from the middle of the base to the vertical angle,
x for half the base, and
y for half the perpendicular;

Then $(2y)^2 = b^2 - x^2$; or $y^2 = \frac{b^2 - x^2}{4}$.

Also $a^2 = \frac{b^2 - x^2}{4} + 4x^2$, or $4a^2 - b^2 = 15x^2$, that is,

$x = \sqrt{\frac{4a^2 - b^2}{15}}$, hence the *base of the triangle* $= 2\sqrt{\frac{4a^2 - b^2}{15}}$

And $y^2 = \frac{b^2 - x^2}{4} = \frac{16b^2 - 4a^2}{4 \times 15}$, or $y = \sqrt{\frac{4b^2 - a^2}{15}}$. Wherefore

$2y = 2\sqrt{\frac{4b^2 - a^2}{15}}$ = *the perpendicular*. Lastly,

$\sqrt{\frac{12a^2 + 12b^2}{15}}$ = *the hypothenuse*. Q. E. I.

PROBLEM XIII.

Put a for the radius of the inscribed circle,
p for the perimeter of the triangle,
x for half the sum of the base and perpendicular, and
y for half their difference:

Then $p-2x$=the hypothenuse, and $\frac{ap}{2}$=the area of the triangle.

But $x^2-y^2=ap$, and $(x+y)^2+(x-y)^2$ being equal to $(p-2x)^2$ or $p^2-4px+4x^2$, it follows that, $p^2-4px+4x^2=2x^2+2y^2$.

And $2\times(x^2-y^2)=2\times ap$. Therefore by addition,

$$\begin{array}{l}\left\{\begin{array}{llll}2x^2-2y^2= & & 2ap & \\ 2x^2+2y^2= & p^2 & * & -4px+4x^2\end{array}\right\}\\ \hline 4x^2 \quad * \quad =p^2+2ap-4px+4x^2,\ \text{or}\end{array}$$

$p^2+2ap-4px=0$. Hence $x=\frac{p^2+2ap}{4p}=\frac{1}{2}(\frac{1}{2}p+a)$

And $y=\sqrt{x^2-ap}=\sqrt{\frac{1}{4}(\frac{1}{2}p+a)^2-ap}$. Therefore

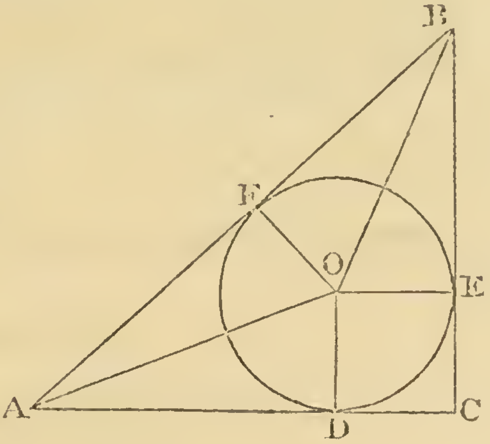
$$\left.\begin{array}{l}\frac{1}{2}(\frac{1}{2}p+a)+\sqrt{\frac{1}{4}(\frac{1}{2}p+a)^2-ap}=\textit{the base or perpendicular}\\ \frac{1}{2}(\frac{1}{2}p+a)-\sqrt{\frac{1}{4}(\frac{1}{2}p+a)^2-ap}=\textit{the perpendicular or base}\end{array}\right\}\text{Q. E. I.}$$

Otherwise.

Let ABC be the given triangle, DEF the inscribed circle, whereof OD, OE, OF are three radii at right angles to the sides of the triangle, AF is equal to AD, and BF to BE: also EC and CD are equal.

Put p for the perimeter,
r for the given radius.
x for EB, and
y for AD.

Then AC$=r+y$; BC$=r+x$; and AB$=x+y$.



But $x+y+(r+y)+(r+x)=p$, or $x+y=\frac{p-2r}{2}=\frac{1}{2}p-r$, and

$$(x+y)^2=(r+y)^2+(r+x)^2;\ \text{or}\ r(x+y)=xy-r^2.$$

In the 1st equation $y=\frac{1}{2}p-r-x$; substitute, therefore, this value for y in the 2d equation, and reduce.

$$x^2-(\tfrac{1}{2}p-r)x=-\tfrac{1}{2}rp.\quad\text{Wherefore}$$

$x=\frac{1}{2}(\frac{1}{2}p-r)\pm\sqrt{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp}$. And

$y=\frac{1}{2}(\frac{1}{2}p-r)\mp\sqrt{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp}$. Hence, by adding r to each,

$$\left.\begin{array}{l}\text{BC}=\frac{1}{2}(\frac{1}{2}p+r)\pm\sqrt{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp}\\ \text{AC}=\frac{1}{2}(\frac{1}{2}p+r)\mp\sqrt{\frac{1}{4}(\frac{1}{2}p-r)^2-\frac{1}{2}rp}\end{array}\right\}\text{Q. E. I.}$$

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PROBLEM XIV.

Let a represent the base,
x one of the segments of the base by the perpendicular,
r the side of the triangle adjacent to x,
s the other side of the triangle,
b the perpendicular; and
$m : n$ the ratio of $r : s$.

Then $a-x$ is the other segment of the base. Also $x^2+b^2=r^2$, and $x^2+b^2+a^2-2ax=s^2$.

But $m^2 : n^2 :: (x^2+b^2) : (x^2+b^2+a^2-2ax.)$ Wherefore $(m^2-n^2)x^2-2am^2x=(n^2-m^2)b^2-a^2m^2$. That is,

$$x^2-\frac{2am^2}{m^2-n^2}x=-b^2-\frac{a^2m^2}{m^2-n^2},$$ which equation resolved,

$$x=\frac{am^2}{m^2-n^2}\pm\sqrt{\frac{a^2m^4}{(m^2-n^2)^2}-\frac{a^2m^2}{m^2-n^2}-b^2}.$$

$$r=\sqrt{b^2+\left(\frac{am^2}{m^2-n^2}\pm\sqrt{\frac{a^2m^4}{(m^2-n^2)^2}-\frac{a^2m^2}{m^2-n^2}-b^2}\right)^2}.$$ And

$$s=\frac{n}{m}\sqrt{b^2+\left(\frac{am^2}{m^2-n^2}\pm\sqrt{\frac{a^2m^4}{(m^2-n^2)^2}-\frac{a^2m^2}{m^2-n^2}-b^2}\right)^2}.$$ *Consequently the triangle is determined, which was to be done.*

PROBLEM XV.

Put a for the hypothenuse,
c for the side of the given square,
x for the perpendicular, and
y for the base of the triangle.

Then As $x : y :: (x-c) : c$, or $y(x-c)=cx$, that is $xy=c(x+y)$

But $a^2=x^2+y^2$. Add $2xy$ to both sides of the equation,

$$a^2+2xy=x^2+2xy+y^2.$$

Substitute $2c(x+y)$ for $2xy$ on the 1st side,

$$a^2+2c(x+y)=x^2+2xy+y^2,$$ or

$$a^2=(x+y)^2-2c(x+y).$$ Therefore

$$x+y=c\pm\sqrt{a^2+c^2},$$ and

$$y=c-x\pm\sqrt{a^2+c^2}.$$

Substitute this value of y in the equation $xy=c(x+y)$.

$$x(c-x\pm\sqrt{a^2+c^2})=c(c\pm\sqrt{a^2+c^2}).$$

That is, $x^2-(c\pm\sqrt{a^2+c^2})x=-c(c\pm\sqrt{a^2+c^2})$.*

Which equation resolved,

$$x=\tfrac{1}{2}(c\pm\sqrt{a^2+c^2})\pm\sqrt{\tfrac{1}{4}a^2-\tfrac{1}{2}c^2\mp\tfrac{1}{2}c\sqrt{a^2+c^2}}=\textit{the perpend.}$$

And

$$y=\tfrac{1}{2}(c\pm\sqrt{a^2+c^2})\mp\sqrt{\tfrac{1}{4}a^2-\tfrac{1}{2}c^2\mp\tfrac{1}{2}c\sqrt{a^2+c^2}}=\textit{the base.}$$

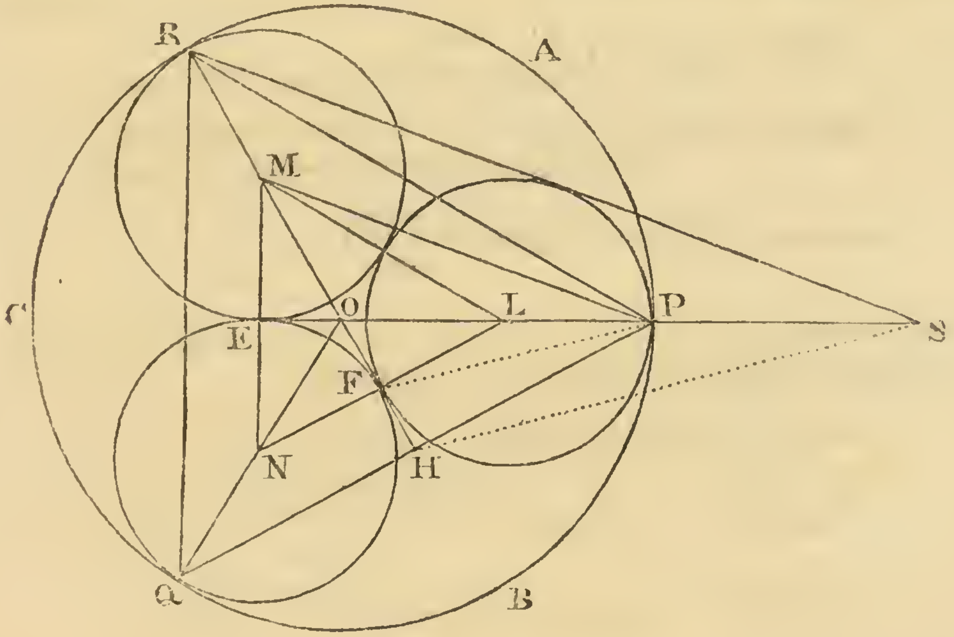
But a plane right angled triangle is determined when the base and perpendicular are known; *therefore*, &c.

---

* Because (line 11 of the Solution) the value of $x+y$ is $c\pm\sqrt{a^2+c^2}$.

PROBLEM XVI.

Let ABC be the given circle, whereof O is the center. Inscribe in it the equilateral triangle PQR (Eucl. iv. 15); join OQ, OR, OP; and produce OP till PS equal the half of PR, or of PQ. Draw SR, and parallel to SR through the point P draw PM meeting OR in M; and through M draw ML parallel to RP; through L, LN parallel to PQ; and join MN.



L, M, N are the centers of three circles that shall touch one another, and the circumference of the given circle ABC.

For bisect PQ in H, and join SH; and parallel to SH draw PF.

(Key to Vol. I. page 364.)

Because POQ is an isoceles triangle, and that, LN is parallel to the base, PL is equal to QN: And because SH, PF are parallel, (PH, LF being also parallel,) and that, SP, PH are equal, PL, LF are equal. In the same manner it may be proved that QN is equal to NF, to NE, to EM, to MR, &c. Moreover it is evident that, NF, FL are in the same straight line.

Putting, therefore, a=radius of the given circle, and
x=radius of one of the inscribed circles,

It is, (*because* NF$=\frac{1}{2}$MN, *and, by similar triangles* OE$=\frac{1}{2}$ OM)

$$(a-x)^2-\left(\frac{a-x}{2}\right)^2=x^2. \quad \text{Whence}$$

$$x=-3a\pm\sqrt{12a^2}=2a\sqrt{3}-3a. \quad \textit{Which was required.}$$

PROBLEM XVII.

Put p for the perimeter of the triangle.
a for the perpendicular falling on the hypothenuse,
x for the greater, } if unequal.
y for the less side, }
Also z for the hypothenuse.

$$\text{Then } z=p-x-y. \quad \text{And}$$

$$z^2=x^2+y^2=p^2-2p(x+y)+x^2+2xy+y^2.$$

That is, $p(x+y)-\frac{1}{2}p^2=xy$; and by comparison,

$$(a+p)\times(x+y)=ap+\tfrac{1}{2}p^2. \quad \text{That is,}$$

$$x+y=p\left(\frac{a+\frac{1}{2}p}{a+p}\right), \text{ and, } y=p\left(\frac{a+\frac{1}{2}p}{a+p}\right)-x.$$

By substituting these values for $(x+y)$, and for y, in the foregoing equations, the following equation is obtained:

$$(a+p)x-p(a+\tfrac{1}{2}p)x=\tfrac{1}{2}ap^2. \quad \text{Hence}$$

$$x=p\left(\frac{a+\frac{1}{2}p}{2a+2p}\right)\pm\frac{p}{2(a+p)}\sqrt{(a-\tfrac{1}{2}p)^2-za^2}. \quad \text{And}$$

$$y=\frac{p(a+\frac{1}{2}p)}{2a+2p}\mp\frac{p}{2a+2p}\sqrt{(a-\tfrac{1}{2}p)^2-2a^2}. \quad \text{Also}$$

$$z=p-x-y=\frac{p^2}{2a+2p}.$$ *Wherefore the triangle is determined.*

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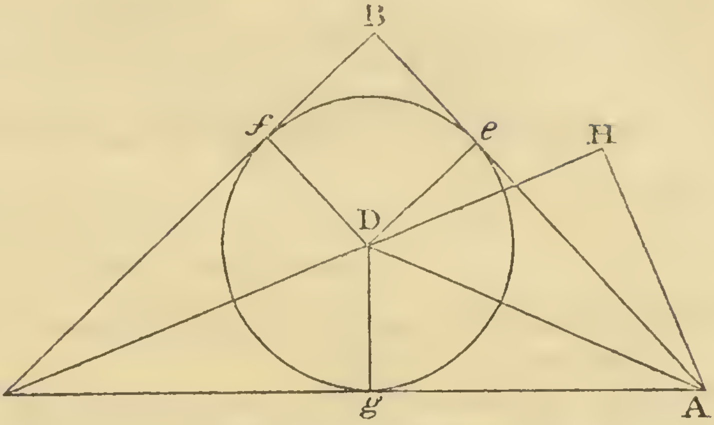
PROBLEM XVIII.



In the annexed fig. let CD be the greater, and AD the less of the two lines of which the difference is given, and let DH be a production of CD, and AH perpendicular to CH, AH is equal to HD, because the angle ADH is equal to the sum of the angles ACD, CAD, together equal to half a right angle, and the angle at H a right angle.

If, therefore, a be put for AC the hypothenuse,
x for CD,
y for AD,
b for the difference of x and y,
r for DH, and
s for AH,

Then (Theor. 4.) $r=s=\frac{y}{\sqrt{2}}$.

But $x^2+y^2+2rx=a^2=x^2+y^2+xy\sqrt{2}$.

Now substituting $x-b$ for y, and c for the $\sqrt{2}$, it will be $x^2-2bx+b^2+x^2+cx^2-cbx=a^2$, that is, $(2+c)x^2-(2+c)bx+b^2=a^2$. Whence

$x=\frac{b}{2}\pm\sqrt{\frac{a^2-b^2}{2+c}+\frac{1}{4}b^2}$, and

$y=\sqrt{\frac{a^2-b^2}{2+c}+\frac{1}{4}b^2}-\frac{1}{2}b$. Consequently the radius of the inscribed circle is known, *and the triangle determined,*

For put $m=\frac{1}{2}b\pm\sqrt{\frac{a^2-b^2}{2+c}+\frac{1}{4}b^2}=$ CD.

$n=\sqrt{\frac{a^2-b^2}{2+c}+\frac{1}{4}b^2}-\frac{1}{2}b=$ AD.

And let De, Df, Dg, be three radii at right angles to the sides of the triangle; likewise put w for Ag, and z for cg.

$z^2-w^2=m^2-n^2$

Also $z+w=a$; and by division

$$z-w=\frac{m^2-n^2}{a}, \text{ that is, } z=\frac{a^2+m^2-n^2}{2a}; \; w=\frac{a^2-m^2+n^2}{2a},$$

And DG $=\sqrt{m^2-\frac{(a^2+m^2-n^2)^2}{4a^2}}$ = *the radius of the inscribed circle.*

PROBLEM XIX.

Put a for half the base, d for the difference of the two sides,
b for the perpendicular,
x for the excess of the greater segment of the base above a.
y for the greater side of the triangle, and
z for the less.

Then $y^2=b^2+(a+x)^2$, or $y=\sqrt{b^2+(a+x)^2}$. Also
$z^2=b^2+(a-x)^2$, or $z=\sqrt{b^2+(a-x)^2}$.

But $\sqrt{b^2+(a+x)^2}-d=\sqrt{b^2+(a-x)^2}$. Hence
$b^2+(a+x)^2-2d\sqrt{b^2+(a+x)^2}+d^2=b^2+(a-x)^2$: and by red.
$4ax+d^2=2d\sqrt{b^2+(a+x)^2}$, or $16a^2x^2+8ad^2x+d^4=4d^2(b^2+a^2+2ax+x^2)$.

Consequently $x=\sqrt{\frac{4d^2(a^2+b^2)-d^4}{16a^2-4d^2}}$, which value put equal c.

$$\left.\begin{array}{l} y=\sqrt{(a+c)^2+b^2}. \\ z=\sqrt{(a-c)^2+b^2}. \end{array}\right\} \text{which were to be determined.}$$

PROBLEM XX.

Let a represent the perpendicular,
b half the base,
c the product of the two sides,
x the excess of the greater segment of the base by the perp. above b;
And put y for the greater side of the triangle,
z for the less.

Then $y=\sqrt{a^2+(b+x)^2}=\frac{c}{z}$

$$z=\sqrt{a^2+(b-x)^2}=\frac{c}{y}=\frac{c}{\sqrt{a^2+(b+x)^2}}, \text{ (or squared)}$$

$$a^2+(b-x)^2=\frac{c^2}{a^2+(b+x)^2}. \text{ Hence}$$

$c^2=(a^2+b^2)^2+x^4+2(a^2-b^2)x^2$, that is, $x^4+2(a^2-b^2)x^2=(a^2+b^2)^2-c^2$.

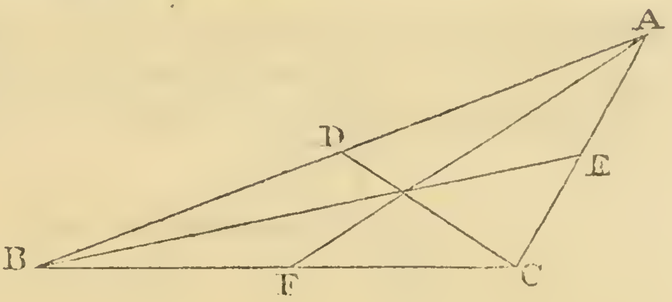
And, this equation resolved,

$$x^2=-(a^2-b^2)\pm\sqrt{(a^2+b^2)^2-c^2+(a^2-b^2)^2}.$$

or $x=\sqrt{(b^2-a^2)\pm\sqrt{(a^2+b^2)^2-c^2+(a^2-b^2)^2}}$, which value put $=d$.

$$\left.\begin{array}{l} y=\sqrt{a^2+(b+d)^2}. \\ z=\sqrt{a^2+(b-d)^2}. \end{array}\right\}\textit{which were to be determined.}$$

PROBLEM XXI.



Let ABC be the triangle, and AF, BE, and CD the three given lines.

If a represent AF,
b, CD. And
c, BE; also
If x be put for BC,
y for AB, and
z for AC;

Then (Theor. 38) $y^2+z^2=2a^2+\frac{1}{2}x^2$, that is,

$y^2+z^2-\frac{1}{2}x^2=2a^2$. For the same reason

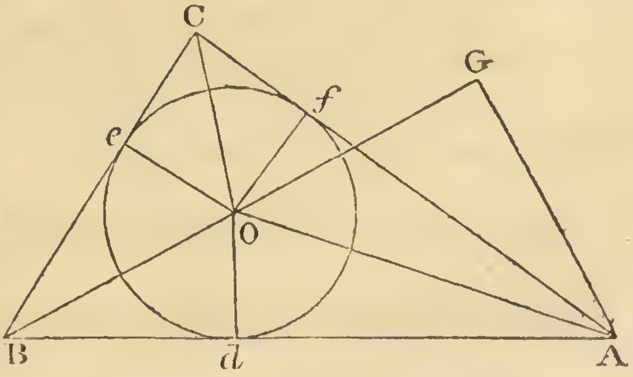
$x^2+z^2-\frac{1}{2}y^2=2b^2$. And

$x^2+y^2-\frac{1}{2}z^2=2c^2$. Comparing the 1st equation with double the sum of the 2d and 3d.

$4\frac{1}{2}x^2=2(2b^2+2c^2-a^2)$ or, divided by $4\frac{1}{2}$, and the square root extracted,

$$\left.\begin{array}{l} x=\frac{2}{3}\sqrt{2b^2+2c^2-a^2}. \\ y=\frac{2}{3}\sqrt{2a^2+2c^2-b^2}. \\ z=\frac{2}{3}\sqrt{2a^2+2b^2-c^2}. \end{array}\right\}\text{Q. E. I.}$$

PROBLEM XXII.



Conceiving the figure constructed (as in the accompanying diagram) draw lines from the center to each of the angles, and to the points of contact. Produce either of the lines joining the angular points and center (as BO) indefinitely through the opposite side, and on it produced let fall a perpendicular (as AG) from that angular point from which the perpendicular falls without the triangle.

Put a for Ad,
b for dB, and
c for Cf. Also
x for Od=Oe=Of.

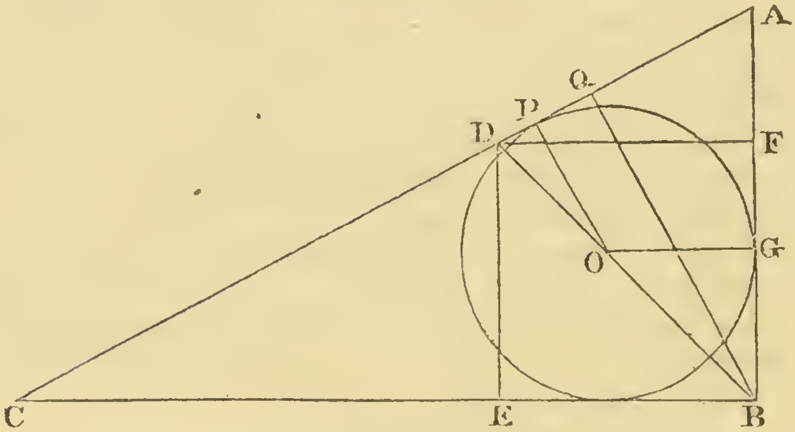
Then $\sqrt{b^2+x^2}$ [BO] : x [Od] : : $a+b$ [AB] : $\frac{ax+bx}{\sqrt{b^2+x^2}}$ [AG].

But BO : Bd : : AB : BG. that is, $\frac{ab+b^2}{\sqrt{b^2+x^2}}-\sqrt{b^2+x^2}=\frac{ab-x^2}{\sqrt{b^2+x^2}}$.

And AG : OG : : Cf : Of, whence $ax+bx : ab-x^2 : : c : x$.

Therefore $x=\sqrt{\frac{abc}{a+b+c}}$. *Which was required.*

PROBLEM XXIII.



Let ABC be the proposed triangle, BFDE the inscribed square, OG and OP radii of the inscribed circle at right angles to AB and AC, and BD a diagonal of the inscribed square; also let BQ be perpendicular to AC from the right angle. Put

a for the side of the given square,
b for the radius of the given circle, and
x for the segment AQ of the base AC by the perpend. BQ,

Then FG$=a-b$ since GB=OG; and

$$a-b : a : : b : \frac{ab}{a-b}=\text{BQ}$$ (*because* GF : BF : : OD : BD : : OP : BQ).

Wherefore (since BD $=\sqrt{2a^2}$), DQ $=\sqrt{2a^2-\frac{a^2b^2}{(a-b)^2}}$. Let this value of DQ be recognized in c, and put d for $\frac{ab}{a-b}$. It is

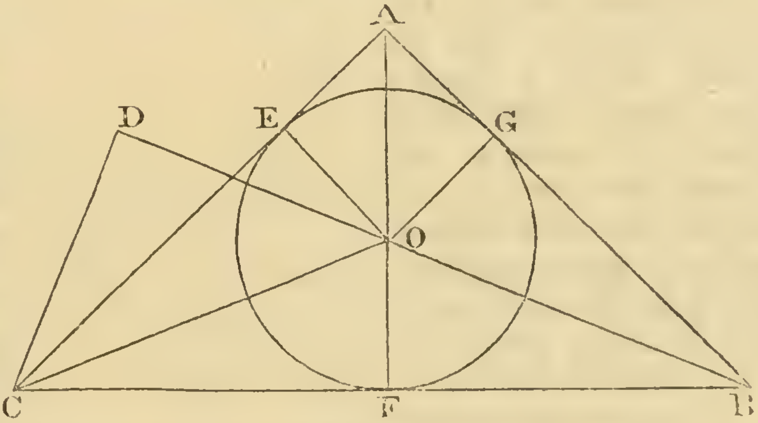
As $x : d :: d : \frac{d^2}{x} =$ CQ [*Theor.* 87]. And $x+c : \frac{d^2}{x} - c :: x : d$.*

Hence $dx+dc=d^2-cx$. That is $x=\frac{d^2-dc}{d+c}=$ AQ, *wherefore the triangle is determined.*

---

* *For the triangles* AFD, DEC *are similar; also the triangle* AQB *is similar to* AFD, *and consequently to* DEC. *Therefore* AD : DC :: AF : DE (= DF) :: AQ : QB.

PROBLEM XXIV.



Having constructed the figure, draw lines from the center of the circle to the points of contact, as OG, OF, OE. Also produce one of the given lines (as BO) indefinitely beyond the center, and on it produced, from either of the other angles let fall a perpendicular (as CD). Then because the angles ABC, BCA, and CAB are together equal to two right angles, the angles OBC, OCB, and OAC are together equal to half of two right angles. But the angle OBC together with the angle OCB, is equal to the angle COD, therefore the angles COD, OAE are equal to the angles COD and OCD, each to each, and either pair to a right angle. Put $a=$ BO

$b=$ AO

$c=$ CO, and

put x for the radius of the inscribed circle.

Then because AO : OE :: CO : OD, $b : x :: c : \frac{c}{b}x=$ OD.

Therefore CD$=\sqrt{c^2-\frac{c^2x^2}{b^2}}=\frac{c}{b}\sqrt{b^2-x^2}$. But, (Theor. 36)

BC$^2=$ BO$^2+$ CO$^2+2$(BO × OD), or AC$=\sqrt{\frac{b(a^2+c^2)+2acx}{b}}$.

(Key to Vol. I. page 365.)

Now BC : CD : : BO : OF. That is, $\sqrt{\frac{b(a^2+c^2)+2acx}{b}} : \frac{c}{b}\sqrt{b^2-x^2}$ $: : a : x$. Hence $bx^2\ (b\times\overline{a^2+c^2}+2acx)=a^2c^2(b^2-x^2)$, and by reduction, $x^3+(\frac{a^2b^2+a^2c^2+b^2c^2}{2abc})x^2=\frac{1}{2}abc$, an equation in which x is determinable, and, x known, *the triangle is determined.*

The equation, however, arising from this problem, as it contains three dimensions of the unknown quantity, admits of *no Geometrical construction by means of a* CIRCLE *and* RIGHT LINES.

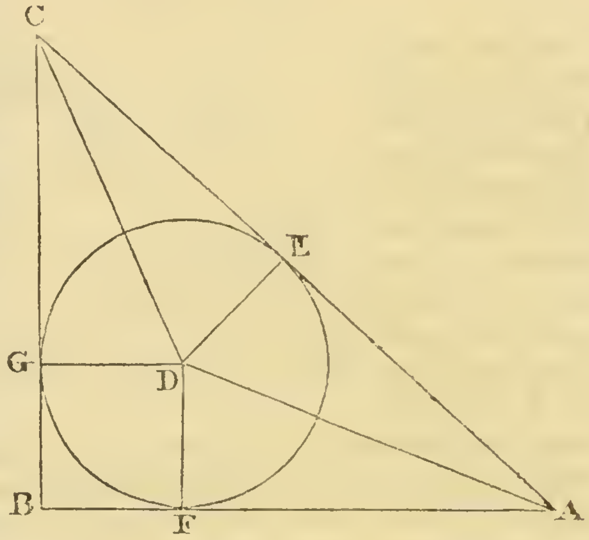
PROBLEM XXV.

Construct a right angled triangle, and inscribe in it a circle; also draw right lines from the center to the points of contact, and to the two acute angles, as in the figure ABCDEFG. Put x for the base (AB),

y for the perpendicular (BC) and let

a represent the given radius (DG, DE, or DF),

b the hypothenuse (AC).



Then, because GB and DF are equal, $y-a$ is the expression for CG; and $x-a=$AF. Also CE and CG are equal [Eucl. i. 26]; and AF is equal to AE.

But AE+CE=AC; that is, $(y-a)+(x-a)=b$, or $x+y=2a+b$.

Now $x^2+y^2=b^2$; comparing, therefore, the double of this equation, with the square of the preceding,

$x^2-2xy+y^2=b^2-4ab-4a^2$. Hence

$x\ -y=\sqrt{b^2-4ab-4a^2}$, consequently

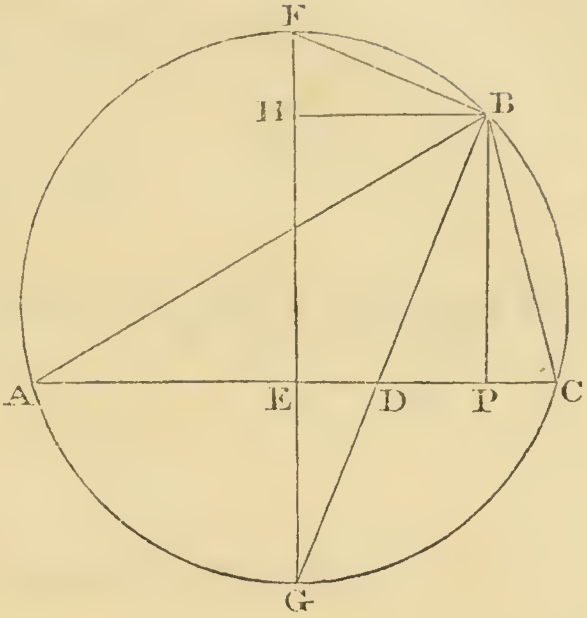
$x\ =a+\frac{1}{2}b\pm\frac{1}{2}\sqrt{b^2-4ab-4a^2}$, and

$y\ =a+\frac{1}{2}b\mp\frac{1}{2}\sqrt{b^2-4ab-4a^2}$.

Wherefore the triangle is determined.

(Key to Vol. I. page 365.)

PROBLEM XXVI.



Let ABC be the triangle, and BD the line bisecting the vertical angle.

Draw the diameter FG at right angles to the base. If BD be produced it will meet the circumference in G, because equal angles stand on equal arcs. Put

a for BD,
b for AE or EC,
c for FG,
And x for EG.

Then [Theor. 61] $x\,(c-x)=b^2$, hence $x=\frac{1}{2}c\pm\sqrt{\frac{1}{4}c^2-b^2}$; for this value of x put e; and join BF, also let y represent DG.

The angle GBF being [Theor. 52] a right angle, the triangles GDE, GFB, are [Theor. 84] similar; therefore $y:e::c:a+y$, that is, $ay+y^2=ec$, hence $y=\pm\sqrt{ec+\frac{1}{4}a^2}-\frac{1}{2}a$. Put f for this value of y; and $z=$ DC.

Then [Theor. 61] $af=2bz-z^2$, that is, $z=b\pm\sqrt{b^2-af}$. Put this value of $z=g$, and from B draw BH at right angles to FG.

GD : GB : : DE : BH, that is,

As $f:(a+f)::(b-g):(b-g+\frac{ab+ag}{f})=$ BH the distance of the perpendicular from the middle of the base, = EP. Join BP.

GE : EH(= BP) : : GD : DB, that is,

As e : BP : : $f:a$, or

BP $=\frac{ac}{f}=$ the perpendicular of the triangle ABC. But

AB $=\sqrt{\text{AP}^2+\text{PB}^2}=\sqrt{\left\{b+(b-g+\frac{ab+ag}{f})\right\}^2+p^2}$. [*where p represents* PB]; and BC $=\sqrt{\text{PC}^2+\text{BP}^2}=\sqrt{\left\{b-(b-g+\frac{ab+ag}{f})\right\}^2+p^2}$.

Wherefore the triangle is determined, which was to be done.

END OF VOL. I.

KEY

TO

HUTTON's COURSE OF MATHEMATICS.

PLANE TRIGONOMETRY.

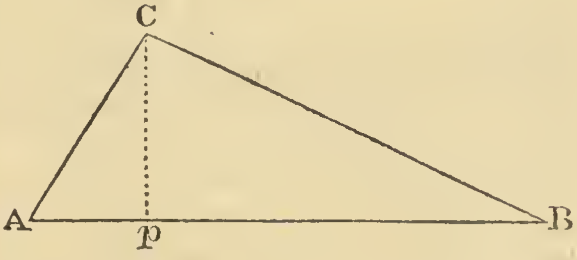
(Key to Vol. II. page 9.)

THEOREM I.

Ex. 2.

1. *Geometrically.*

Draw an indefinite line, and take in it AB 365 *poles* from any convenient scale of equal parts. At A make the angle BAC 57° .. 12′, and at B make the angle ABC 24° .. 45′, ABC is the triangle in species. But



$$\left.\begin{array}{lll} \text{AC measures} & 154\frac{1}{3} & \text{poles} \\ \text{BC - - - - - -} & 310 & \text{poles} \end{array}\right\} \text{on the scale employed for AB.}$$

And the angle ACB by a line of Chords or Goniometer* is 98° .. 3′.

* *Any Instrument to measure Angles by Inspection.*

2. *Arithmetically.*

$180° - \left\{ 57° .. 12' \,(=\angle\text{A}) + 24° .. 45' \,(=\angle\text{B}) \right\} = 98° .. 3'$ the angle C.

And

As Sin. 98° .. 3′ ∠C	- - -	Log. ar. co. –	10·004301
: Sin. 24° .. 45′ ∠B	- - - - - - - -	Log.	9·621861
: : 365 poles AB	- - - - - - - -	Log.	2·562293
: 154·33 *poles* AC		*Log.*	2·188455

Also,

As Sin. 98° .. 3′ ∠C	- - -	Log. ar. co. –	10·004301
: Sin. 57° .. 12′ ∠A	- - - - - - - -	Log.	9·924572
: : 365 poles AB	- - - - - - - -	Log.	2·562293
: 309·86 *poles* BC		*Log.*	2·491166

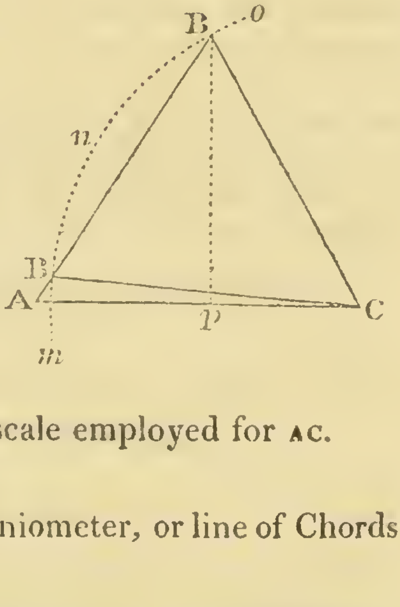
3. *Instrumentally.*

The sine of 98° .. 3′ = sine 81° .. 57′, and the extent from 81° .. 57′ to 24° .. 45′ on the line of sines, reaches from 365 to 154⅓ on the line of numbers, for the length of the side AC in poles. Likewise the extent from 81° .. 57′ to 57° .. 12′ on the line of sines reaches from 365 to 310 *nearly*, on the line of numbers, for the length of BC in poles.

(Page 10.)

Ex. 3. 1. *Geometrically.*

Draw an indefinite line, and take in it AC 120 *feet*. At A make the angle CAB 57° .. 28′, and with a radius of 112 from the scale on which AC measures 120, describe the arc *mno* cutting AB in the points B and B, either of the triangles ABC answers the conditions, shewing that the case is ambiguous. Now



AB measures $\left\{\begin{matrix}16\frac{1}{2}\\ 112\frac{1}{2}\end{matrix}\right\}$ feet, on the scale employed for AC.

The angle B $\left\{\begin{matrix}115° .. 25′\\ 64° .. 35′\end{matrix}\right\}$ by a Goniometer, or line of Chords.

And the angle C $\left\{\begin{matrix}7° .. 7′\\ 57° .. 57′\end{matrix}\right\}$ by subtraction, or admeasurement.

(Key to Vol. II. page 10.)

2. *Arithmetically.*

As 112 feet BC - - - - - - Log. ar. co. −3·95078
: 120 feet AC - - - - - - - - - - - Log. 2·07918
:: Sin. 57° .. 28′ ∠A - - - - - - - - Log. 9·92587

: *Sin.* $\left\{\begin{array}{c}64^\circ\,..\,35'\\115^\circ\,..\,25'\end{array}\right\}$ ∠B - - - - *Log.* 9·95583

Hence

The angle C $=\left\{\begin{array}{c}7^\circ\,..\,\;7'\\57^\circ\,..\,57'\end{array}\right\}$ the supplement of ∠(A + B)

And

As Sin. 57° .. 28′ ∠A.
: Sin. $\left\{\begin{array}{c}7^\circ\,..\,\;7'\\57^\circ\,..\,57'\end{array}\right\}$ ∠C.
:: 112 feet, BC.
: $\left\{\begin{array}{c}16{\cdot}47\\112{\cdot}6\end{array}\right\}$ *feet*, AB

3. *Instrumentally.*

The extent from 112 to 120 on the line of numbers reaches from 57° .. 28′ to 64° .. 35′ (=sine of 115° .. 25′) on the line of sines, for the ∠B. Hence ∠C $=\left\{\begin{array}{c}7^\circ\,..\,\;7'\\57^\circ\,..\,57'\end{array}\right\}$

Again the extent from 57° .. 28′ to 7°. 7′, or 57° .. 57′ on the line of sines, reaches from 112 to $16\frac{1}{2}$ or 112·6 on the line of numbers, the two values of AB in feet.

(Page 12.)

THEOREM II.

Ex. 2. [*See the figure in Example 2, Theorem* I.]

1. *Geometrically.*

From a scale of equal parts take AB 365 *poles*. At A make the angle BAC 57° .. 12′, and from A with a radius of 154·33 on the scale on which AB measures 365, determine the point C; lastly join BC. ABC is the triangle required.

BC measures 309·8 on the scale employed for AB.

And

By the Goniometer, or a line of Chords $\left\{\begin{array}{l}\angle B=24^\circ\,..\,45'\\ \angle C=98^\circ\,..\,\ 3'\end{array}\right\}$

2. *Arithmetically.*

As 519·33 AB+AC - - - - - - - - - - - Log. ar. co. −3·2845566
: 210·67 AB−AC - - - - - - - - - - - - - - - Log. 2·3236027
: : Tang. 61° .. 24′, half supp. ∠A - - - - - - Log. 10·2634301

: *Tang.* 36° .. 39′ *half the diff. of the* ∠*les* B *and* C. *Log.* 9·8715894

But 61° .. 24′ + 36° .. 39′ = 98° .. 3′ the angle C.
And 61° .. 24′ − 36° .. 39′ = 24° .. 45′ the angle B.

Again,

As Sin. 98° .. 3′ ∠C - - - Log. ar. co. −10·004301
: Sin. 57° .. 12′ ∠A - - - - - - - - Log. 9·924572
: : 365 poles AB - - - - - - - - Log. 2·562293

: 309·86 *poles* BC - - - - - - - - *Log.* 2·491166

3. *Instrumentally.*

The extent from $519\frac{1}{3}$ to 210·7 on the line of numbers reaches from 61° .. 24′ beyond 45° on the line of tangents; but this excess above 45° applied backwards from 45° reaches to $36^\circ\frac{2}{3}$ *half the diff. of the* ∠*les* B *and* C. Hence C = 98° .. 3′ and B = 24° .. 45′. Again the extent from 81° .. 57′ to 57° .. 12′ on the line of sines reaches from 365 to 309·8 on the line of numbers, for the length of BC in poles.

Ex. 3. [*See the Greater triangle* ABC *in the figure to Example* 3, *Theorem* I.]

1. *Geometrically.*

Take AC 120 *yards* from any convenient scale of equal parts, and at C make the angle ACB 57° .. 57′; then from C with a radius of 112 from the scale on which AC measures 120, determine B; lastly join AB.

AB measures 112·6 on the scale employed for AC.

And

By the Goniometer, or a line of Chords $\left\{\begin{array}{l}\angle A=57^\circ\,..\,28'\\ \angle B=64^\circ\,..\,35'\end{array}\right\}$

(Key to Vol. II. page 12.)

2. *Arithmetically.*

As 232 yards AC+BC - - - - - - - - - Log. ar. co. −3·6345120
: 8 yards AC−BC - - - - - - - - - - - - - - - Log. 0·9030900
: : Tang. 61° .. 1′ .. 30″ half supp. ∠C - - - - Log. 10·2566460

: *Tang.* 3° .. 34′ *half the diff. of the* $\angle^{les}$ A *and* B. *Log.* 8·7942480

$$\left.\begin{array}{l} \text{But } 61^\circ\,..\,1'\tfrac{1}{2}+3^\circ\,..\,34'=64^\circ\,..\,35'\tfrac{1}{2}\ \angle\text{B.} \\ \text{And } 61^\circ\,..\,1'\tfrac{1}{2}-3^\circ\,..\,34'=57^\circ\,..\,27'\tfrac{1}{2}\ \angle\text{A.} \end{array}\right\}*$$

Again,

As Sin. $57^\circ\,..\,27'\tfrac{1}{2}$ ∠A - - - Log. ar. co. −10·0741722
: Sin. 57° .. 57′ ∠C - - - - - - - - Log. 9·9281834
: : 112 yards BC - - - - - - - - - - - Log. 2·0492180

: 112·61 *yards* AB - - - - - - - - - *Log.* 2·0515736

3. *Instrumentally.*

The extent* from 232 to 8 on the line of numbers reaches from 61° .. 1′ beyond 45° on the line of tangents; but this excess above 45°, applied backwards from 45°, reaches to 3° .. 34′ *half the diff. of the* $\angle^{les}$ A *and* B.

$$\text{Hence}\left\{\begin{array}{l} \angle\text{B}=64^\circ\,..\,35' \\ \angle\text{C}=57^\circ\,..\,27' \end{array}\right\}$$

Again the extent from 57° .. 27′ to 57° .. 57′ on the line of sines reaches from 112 to 112·6 on the line of numbers, for the length of AB in yards.

---

* *These angles differ half a minute from the angles given in the figure, because the calculation throughout has not been carried to seconds.*

* *It is necessary to employ some multiple of* 232 *and* 8, *instead of the numbers themselves, and such that the multiple of* 8 *may exceed* 100, *if a two-foot scale be employed.*

(Page 14.)

THEOREM III.

Ex. 2. [*See the figure in Example* 2, *Theorem* I.]

(Key to Vol. II. page 14.)

1. *Geometrically.*

From any convenient scale of equal parts take AB 365 *poles,* and with a radius 154·33 from the same scale, and center A, describe an arc; also from B as a center with a radius 309·86 of the scale on which AB measures 365, cut the former arc in C. Join AC and BC.

$$\text{By a line of Chords, or Goniometer} \left\{ \begin{array}{l} \angle A = 57^\circ \,..\, 12' \\ \angle B = 24^\circ \,..\, 45' \\ \angle C = 98^\circ{}^{*} \,..\, 3' \end{array} \right\}$$

* *A typographical error in the Course.*

2. *Arithmetically.*

As 365 poles AB - - - - - - - - - - - - Log. ar. co. −3·4377071
: 469·19 poles AC+BC - - - - - - - - - - - - - Log. 2·6666958
: : 155·53 poles BC−AC - - - - - - - - - - - - - Log. 2·1918142

: 197·8 *poles diff. of the segments of* AB *by a perp. from* C - - - *Log.* 2·2962171

Hence A*p*=83·6 poles, and B*p*=281·4, by ½ *sum* and ½ *diff.* of the segments of the base.

And

As 154·33 pol. AC : 83·6 pol. A*p* : : Sin. 90° ∠A*p*C : *Cos.* 57° .. 12′ ∠A.

As 309·86 pol. BC : 281·4 pol. B*p* : : Sin. 90° ∠B*p*C : *Cos.* 24° .. 45′ ∠B.

Wherefore, by subtraction, ∠C=98° .. 3′.

3. *Instrumentally.*

The extent from 365 to 469·2 reaches from 155½ to 198 *nearly,* on the line of numbers, *for the difference of* A*p and* B*p.*

Hence A*p*=83½ poles, and B*p*=281½ poles.

Again the extent from 154⅓ to 83½ on the line of numbers, reaches on the line of sines from 90° to 32° .. 48′, of which the complement is 57°..12′ the ∠A.

Also the extent from 310 to 281½ on the line of numbers reaches on the line of sines from 90 to 65° .. 15′, of which the complement is 24° .. 45′ the ∠B; and consequently the ∠C= 98° .. 3′ *as before.*

(Key to Vol. II. page 15.)

Ex. 3. [*See the Greater triangle* ABC *in the fig. to Ex.* 3. *Th.* I. *but transpose* B *and* C.]

1. *Geometrically.*

Take AB 120 from any convenient scale of equal parts, and with A as a center at the distance of 112·6 from the same scale describe an arc; lastly cut this arc with 112 described from the center B.

By a line of Chords, or Goniometer $\begin{cases} \angle A = 57^\circ .. 28' \\ \angle B = 57^\circ .. 57' \\ \angle C = 64^\circ .. 35' \end{cases}$

2. *Arithmetically.*

As 120 AB - - - - - - - - - Log. ar. co. –3·9208188
: 224·6 AC+BC - - - - - - - - - - - Log. 2·3514098
: : ·6 AC–BC - - - - - - - - - - - Log. –1·7781513

: 1·123 *diff. of the segments of* AB - *Log.* 0·0503799

Wherefore, *by* ½ *Sum and* ½ *diff.* A*p*=60·566, and B*p*=59·434·

And

As 112·6 AC : 60·566 A*p* : : Sin. 90° ∠A*p*C : *Cos.* 57° .. 28′ ∠A.

Also

As 112 BC : 59·434 B*p* : : Sin. 90° ∠B*p*C : *Cos.* 57° .. 57′ ∠B.
Hence, by subtraction, ∠C=64° .. 35′.

3. *Instrumentally.*

The extent on the line of numbers from 120 to 224·6 will reach from any multiple of ·6 to the same multiple of $1\frac{1}{8}$ *nearly*, the diff. of A*p* and B*p*.

Hence A*p*=$60\frac{9}{16}$, and B*p*=$59\frac{7}{16}$.

Again the extent from 112·6 to $60\frac{9}{16}$ on the line of numbers will reach from 90° to 32° .. 32′ on the line of sines; wherefore ∠A=57° .. 28′ the complement of the 4th term. In like manner,

The extent from 112 to $59\frac{7}{16}$ on the line of numbers will reach on the line of sines from 90° to 32° .. 3′, of which the complement is 57° .. 57′=∠B; and (by Eucl. i. 32.) ∠C=64 .. 35′ which might be proved by a third proportion.

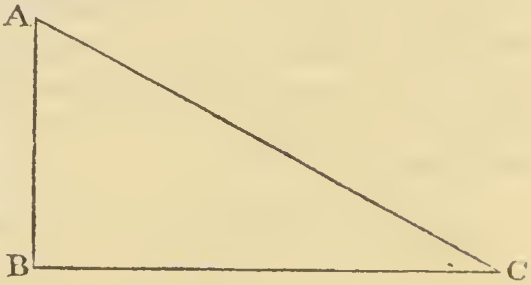
(Key to Vol. II. page 16.)

THEOREM IV.

1. *Geometrically.*

Ex. 2.

Take AB 180 from any scale of equal parts and at A make the angle BAC 62° .. 40′, lastly from B raise the perpendicular BC meeting AC in C.



$\left.\begin{array}{l}AC=392 \\ BC=348{\cdot}2\end{array}\right\}$ on the scale employed for AB.

2. *Arithmetically.*

As Radius AB - - - - - - - - Log. ar. co. – 10·
: Tang. 62° .. 40′ BC - - - - - - - - Log. 10·2866141
: : 180 AB - - - - - - - - - - - - - - Log. 2·2552725

: 348·2464 BC - - - - - - - - - - - - *Log.* 2·5418866

And

As Radius AB - - - - - - - - Log. ar. co. – 10·
: Secant 62° .. 40′ AC - - - - - - - Log. 10·3380298
: : 180 AB - - - - - - - - - - - - - - Log. 2·2552725

: 392·0146 AC - - - - - - - - - - - *Log.* 2·5933023

3. *Instrumentally.*

The extent from 45° to 62° .. 40′ on the line of tangents, reaches from 180 to $348\frac{1}{5}$ on the line of numbers, for *the length of* BC.

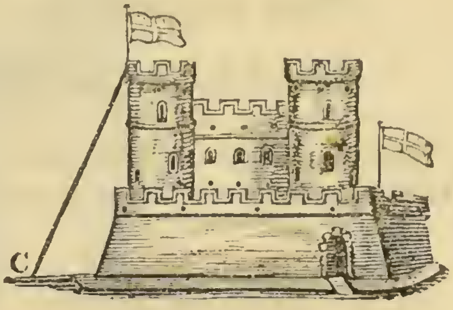
And the extent from 62° .. 40′ to 90° on the line of sines, reaches from $348\frac{1}{5}$ to 392 on the line of numbers for the *length of* AC.

(Key to Vol. II. page 22.)

MENSURATION OF HEIGHTS AND DISTANCES.

⁂ *In the subsequent Examples it has been thought unnecessary to detail the Construction, except in a few intricate cases.*

Ex. 6



Let the $\angle$ c, in the accompanying figure, be the angle of inclination of the ladder to the horizon, and the fort as there represented, the ditch extending or supposed to extend from the foot of the ladder to the base of the perpendicular tower. It is

As Radius - - - - - - - - - Log. ar. co. – 10·
: Tang. 62° .. 40′ $\angle$ c - - - - - - - - - Log. 10·286614
: : 36 feet, the breadth of the ditch, - Log. 1·556303

: 69·649 *feet, the height of the wall.* - *Log.* 1·842917

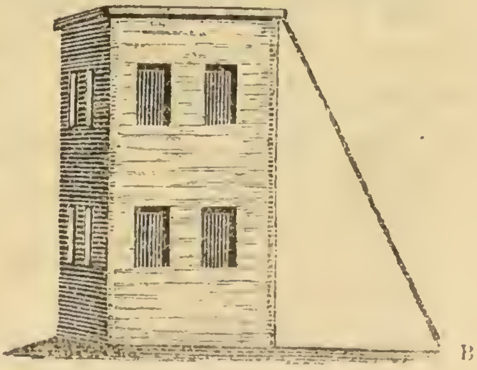
And,
As Sin. 62° .. 40′ $\angle$ c - - - Log. ar. co. – 10·0514158
: Sin. 90° $\angle$ subtended by the ladder, Log. 10· ———
: : 69·649 feet, the height of the wall, Log. 1·8429169

: 78·403 *feet, the length of the ladder. Log.* 1·8943327

Or, instead of the last proportion,

As Rad. : Sec. 62° .. 40′ $\angle$ c : : 36 ft., breadth of the ditch : 78·403 ft., *length of the ladder.*

Ex. 7.



Let B in the annexed diagram be the foot of the shoar, and the building as there represented. It will be

(Key to Vol. II. page 22.)

As 11 feet, the extent of strut, Log. ar. co. − 2·9586073
: 23·8$\dot{3}$ feet, the height of the jamb, Log. 1·3771790
: : Radius - - - - - - - - - - - - - - Log. 10·- - - - - - -

: *Tang.* 65° .. 13′ ∠B - - - - - - - - *Log.* 10·3357863

And,

As Sin. 65° .. 13′ ∠B - - - Log. ar. co. − 10·0419622
: Sin. 90° ∠ subtended by the shoar, Log. 10· ———
: : 23·8$\dot{3}$ feet, the height of the jamb, Log. 1·3771790

: 26·251 *feet, the length of the shoar.* *Log.* 1·4191412

This Question is easily solved by Common Arithmetic, on the Principles deduced from Eucl. i. 47.

Ex. 8. Here are given two right angled triangles, the perpendicular of the one being 33 feet, of the other 21 feet, and the hypothenuse of either, 40 feet, to find the *sum* of the bases.

Now

$\sqrt{40^2-33^2}=\sqrt{1600-1089}=22{\cdot}605$ feet, the base of the 1st triangle.

And

$\sqrt{40^2-21^2}=\sqrt{1600-441}=34{\cdot}044$ feet, the base of the 2d triangle.

But

ft. ft.
22·605 + 34·044 = 56·649 feet, the *required breadth of the street.*

This solution being much shorter than by Sines and Tangents, it would be superfluous to add the Trigonometrical proportions.

Ex. 9. Here are given the base (15 feet) and hypothenuse (39 feet) of a right angled triangle, to find the *sum* of the hypothenuse and perpendicular. Now

$\sqrt{39^2-15^2}=\sqrt{1521-225}=36$ feet, left standing.
Add 39 feet, the part broken off.

Sum 75 feet, *the whole height.*

(Key to Vol. II. page 22.)

Trigonometrically.

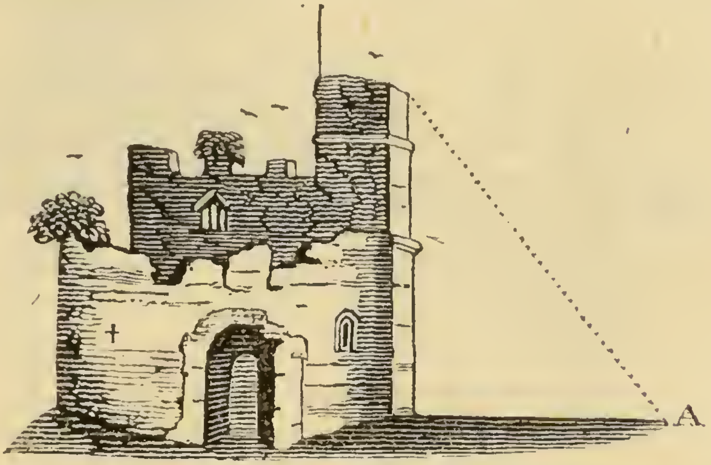
As $\overset{\text{ft.}}{39} : \overset{\text{ft.}}{15}$: : Rad. : Sin. 22° .. 37′ the vertical ∠, and

As Tang. 22° .. 37′ : Rad. : : $\overset{\text{ft.}}{15} : \overset{\text{ft.}}{36}$ left standing.

Hence, as before, *the whole height* 75 *feet.*

Ex. 10.

Let A in the marginal figure be the observed angle 52° .. 30′, and the tower as there represented. It is



As Radius - - - - - - - - - - - - - Log. ar. co. —10· ———
: Tang. 52° .. 30′ ∠A - - - - - - - - - - - Log. 10·1150195
: : 170 feet, the distance of A from the tower, Log. 2·2304489

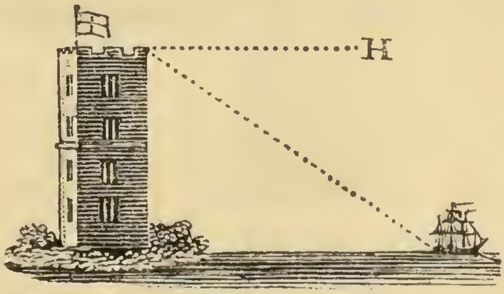
: 221·54 *feet, the altitude required.* - - - - *Log.* 2·3454684

Otherwise.

As Cos. 52° .. 30′ ∠A : Sin. 52° .. 32′ ∠A : : 170 feet : 221·54 *feet,* as before.

Ex. 11.

Let H in the annexed figure be horizontal with the top of the tower, and the ship* as there represented. The height of the tower subtends at the vessel an angle of 35°. [*Eucl.* i. 29.]



Therefore,

As Sin. 35° the complement of the ∠ of depression, } Log. ar. co. —10·2414087
: Cos. 35° (the same angle), - - - - - - - - - Log. 9·9133645
: : 143 feet, the height of the tower, - - - - - Log. 2·1553360

: 204·22 *feet, the distance required.* - - - - - *Log.* 2·3101092

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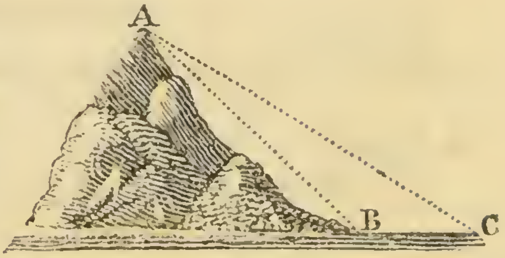
* The Ship is purposely diminished in the diagram to a *Model* at anchor.

(Key to Vol. II. page 23.)

Ex. 12.



Let A be the top of the hill, B and C the two stations of observation. The angle BAC = 15° [Eucl. i. 13 and 32.]

And

As Sin. 15° ∠BAC - - - - - - Log. ar. co. − 10·5870038
: Sin. 31° ∠C - - - - - - - - - - - - Log. 9·7118393
: : 200 yards BC - - - - - - - - - - - - Log. 2·3010300

: 397·99 *yards* AB - - - - - - - - - *Log.* 2·5998731

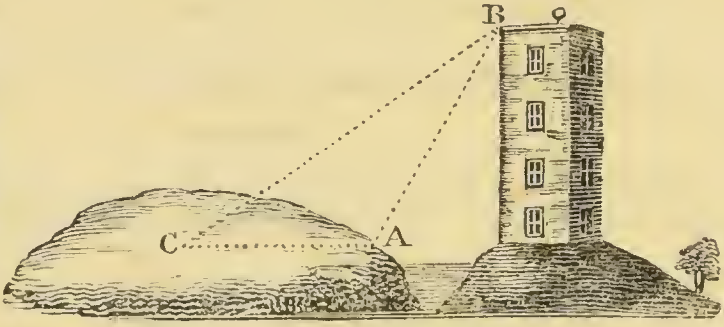
Also

As Radius - - - - - - - - - Log. ar. co. − 10·
: Sin. 46° the observed ∠B - - - - Log. 9·8569341
: : 397·99 yards AB - - - - - - - - - Log. 2·5998731

: 286·29 *yards, the height of the hill.* - *Log.* 2·4568072

Ex. 13.

Let A and C be the two stations, and B the summit of the tower. The angle ABC (for reasons given in the last Ex.) is 26°.



Now

As Sin. 26° ∠ABC - - - - - - Log. ar. co. − 10·3581580
: Sin. 32° ∠ACB - - - - - - - - - - Log. 9·7242097
: : 300 feet AC - - - - - - - - - - - Log. 2·4771213

: 362·65 *feet* AB - - - - - - - - - - *Log.* 2·5594890

And

As Rad. : Sin. 58° obser. ∠A : : 362·65 ft. AB : 307·53 ft. *height of the tower.*

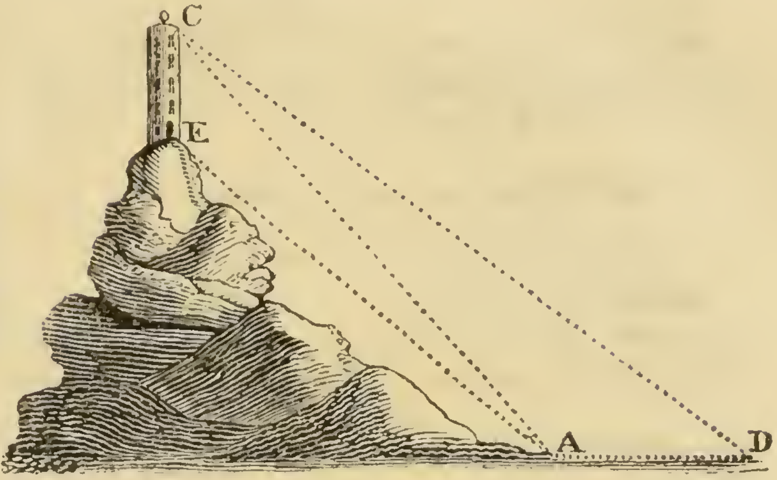
Also

As Sin. 58° : Cos. 58° : : 307·53 feet : 192·15 feet *distance of* A *from the tower.*

(Key to Vol. II. page 23.)

Ex. 14.

Let A and D be the two stations, E the top of the hill, and C the top of the tower. It will be



As Sin. 17° .. 15′ ∠ADC - - Log. ar. co. −10·5279144
: Sin. 33° .. 45′ ∠D - - - - - - - - Log. 9·7447390
: : 200 feet AD - - - - - - - - - - - - Log. 2·3010300

: 374·7 *feet* AC - - - - - - - - - - - - *Log.* 2·5736834

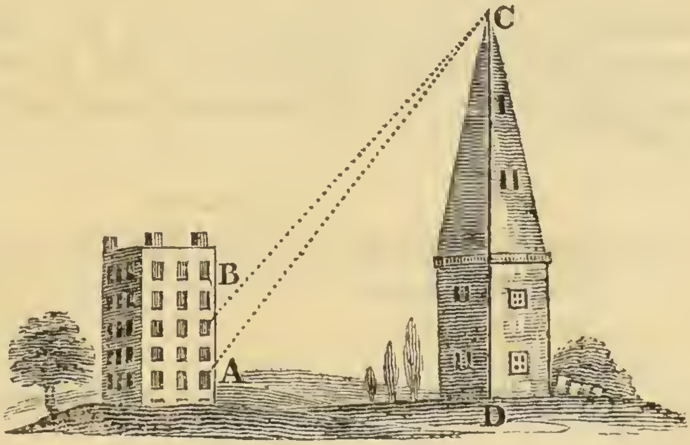
And

As Sin. 130° ∠AEC - - - - Log. ar. co. −10·1157460
: Sin. 11° ∠EAC - - - - - - - - - - Log. 9·2805988
: : 374·7 feet AC - - - - - - - - - - - Log. 2·5736834

: 93·33148 *feet, the height required.* - *Log.* 1·9700282

Ex. 15.

If A and B represent the two windows at which the observations were made, and C the top of the steeple, it will be,



As Sin. 2° .. 30′ ∠ ACB - - - - Log. ar. co. −9·3603204
: Sin. 127° .. 30′ ∠CBA - - - - - - Log. 9·8994667
: : 18 feet AB - - - - - - - - - - - - Log. 1·2552725

: 327·38 *feet* AC - - - - - - - - - - - *Log.* 2·5150596

(Key to Vol. II. page 23.)

And,

As Sin. 90° ∠CDA - - - - - - Log. ar. co. −10·———
: Sin. 40° ∠DAC - - - - - - - - - - - Log. 9·808067
: : 327·38 feet AC - - - - - - - - - - - - Log. 2·515060

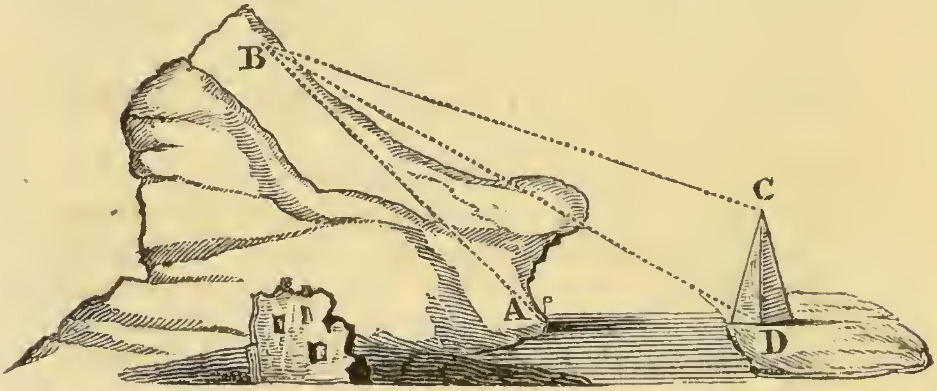
: 210·44 *feet, the height required.* - - - *Log.* 2·323127

Lastly,

As Sin. 90° ∠CDA - - - - - - Log. ar. co. −10·———
: Sin. 50° ∠ACD - - - - - - - - - - - Log. 9·884254
: : 327·38 feet AC - - - - - - - - - - - - Log. 2·515060

: 250·79 *feet, the distance sought.* - - *Log.* 2·399314

Ex. 16.



Let A and B represent the two stations; and CD the object observed. The ∠BDA [Eucl. i. 29.] is 27°; the ∠ABD (by comparison) 15°; the ∠ BAD [Eucl. i. 32.] is 138°; the ∠CBD (by comparison) 8°; and the ∠DCB (*by subtraction, and Eucl.* i. 32.) is 109°.

Wherefore,

As Sin. 27° ∠BDA - - - - - - Log. ar. co. −10·342953
: Sin. 15° ∠ABD - - - - - - - - - - - Log. 9·412996
: : 264 feet AB - - - - - - - - - - - - - - Log. 2·421604

: 150·5 *feet* AD, *distance required.* - - Log. 2·177553

And

As Sin. 27° ∠BDA - - - - - Log. ar. co. −10·342953
: Sin. 138° ∠BAD - - - - - - - - - - - Log. 9·825511
: : 264 feet AB - - - - - - - - - - - - - Log. 2·421604

: 389·1 *feet* BD - - - - - - - - - - - - - *Log.* 2·590068

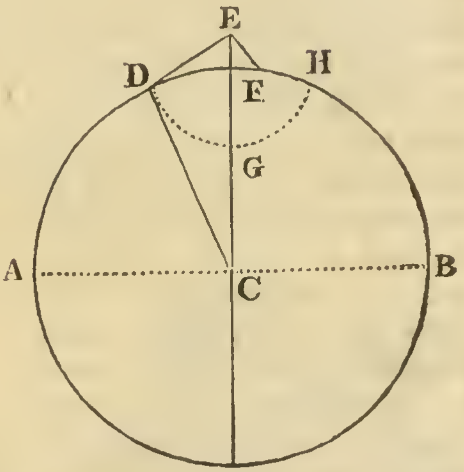
(Key to Vol. II. page 23.)

Lastly.

As Sin. 109° ∠ DCB : Sin. 8° ∠ DBC : : $\overset{\text{feet}}{389{\cdot}1} : \overset{\text{feet}}{57{\cdot}26}$ *height reqd.*

Ex. 17.

If EF be the perpendicular altitude of the mountain, DGH the boundary of the visual horizon, AB the true horizon, and C the center of the earth, it will be



As Co. Sin. 87° .. 58′ ∠ FED - Log. ar. co. —9·4500052
: Sin. 87° .. 58′ *same* ∠ - - - - - - - Log. 9·9997265
: : 2·5 miles FE - - - - - - - - - - - - Log. 0·3979400

: 70·416 *miles, the length of a tangent from* F, *intercepted by* ED. } *Log.* 1·8476717

And,

As Co. Sin. 87° .. 58′ ∠ FED - Log. ar. co. —9·4500052
: Sin. 90° ∠ EF*h** - - - - - - - - - Log. 10· ———
: : 2·5 miles FE - - - - - - - - - - - - Log. 0·3979400

: 70·46 *miles the part of* ED *above the horizont. plane by a tangent at* F. } *Log.* 1·8479452

But $\overset{\text{miles}}{70{\cdot}46} + \overset{\text{miles}}{70{\cdot}416} = 140{\cdot}876$ miles, the visual distance.

And [*Eucl.* III. 36.]

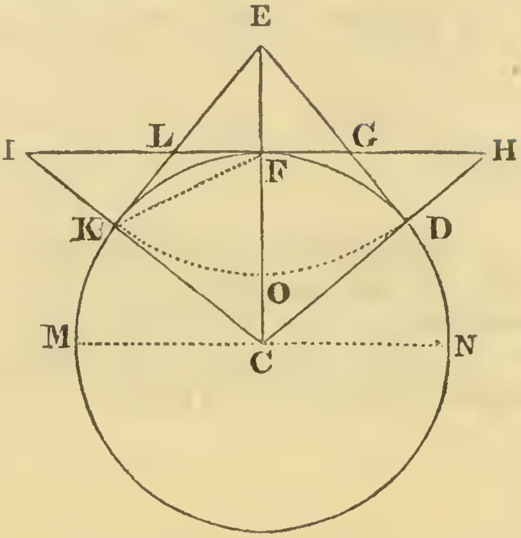
$$\frac{\overline{140{\cdot}876}|^2}{2{\cdot}5} - 2{\cdot}5 = 7936$$ *miles, the diameter of the Earth.*

---

* *Where* EF *denotes the perpendicular, and* F*h* *the horizontal plane by the circular motion of a tangent at* F.

(Key to Vol. II. page 23.)

To render this more plain,



Let MN be the true horizon, KOD the boundary of the visual horizon to E any altitude whatever above F the surface of the Earth, of which the center is C; and let IH be a tangent to the circle at F, parallel to MN, and cutting the tangents ED and EK in the points G and L; EG+GF=ED. For draw CH through the point D, and CI through the point K; also join KF. The three triangles CFH, CDE, and CFI are in all respects equal. But the triangles EFG, HDG, are similar to the triangle EDC, consequently GF is equal to GD, and EG+GF=ED. Or, because CKE, CFI are both right angles, and that CK is equal to CF, the angles LKF, LFK are equal, and [Eucl. i. 6.] KL equal to LF; hence EL+LF=EK=ED. Therefore,

As Rad. EF - - - - - - - - Log. ar. co. −10· ———
: 2·5 miles, *(altitude given)* - - - - Log. 0·3979400
: : Tang. 87°..58′ ∠CED - - - - - Log. 11·4497317

: 70·416 *miles* FG - - - - - - - - - - *Log.* 1·8476717

: : Secant 87°..58′ ∠CED - - - - - Log. 11·4500052

: 70·46 *miles* EG - - - - - - - - - - *Log.* 1·8479452

miles miles miles
But 70·416+70·46=140·876=ED *as before.*
Whence, also, 7936 miles, the diameter of the Earth.

(Page 24.)

Ex. 18. Here are given the three angles of a plane triangle, and one side, to find the other two sides. Now if the fort be denoted by F, and the two ships of war respectively by A and B, it is

(Key to Vol. II. page 24.)

As Sin. 11° ∠AFB - - - - - Log. ar. co. −10·719401
: 440 yards AB - - - - - - - - - - - - Log. 2·643453
: : Sin. 83° .. 45′ ∠ABF - - - - - - - - Log. 9·997411

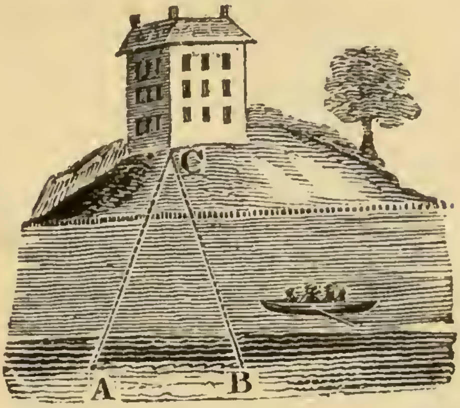
: 2292·26 *yards* AF - - - - - - - - - - *Log.* 3·360265

: : Sin. 85° .. 15′ ∠BAF - - - - - - - - Log. 9·998506
: 2298·05 *yards* BF - - - - - - - - - - *Log.* 3·361360

Therefore { 2292·26 *yards, distance of the* 1*st ship*
2298·05 *yards, distance of the 2d ship* } Ans.

Ex. 19.

Let A and B be the two stations, and C the house. By the question the ∠ABC is 68° .. 2′, the ∠BAC 73° .. 15′, consequently the ∠ACB 38° .. 43′; also AB is 400 yards. But,



As Sin. 38° .. 43′ ∠ACB - - - Log. ar. co. −10·203794
: 400 yards AB - - - - - - - - - - - Log. 2·602060
: : Sin. 68° .. 2′ ∠ABC - - - - - - - - Log. 9·967268

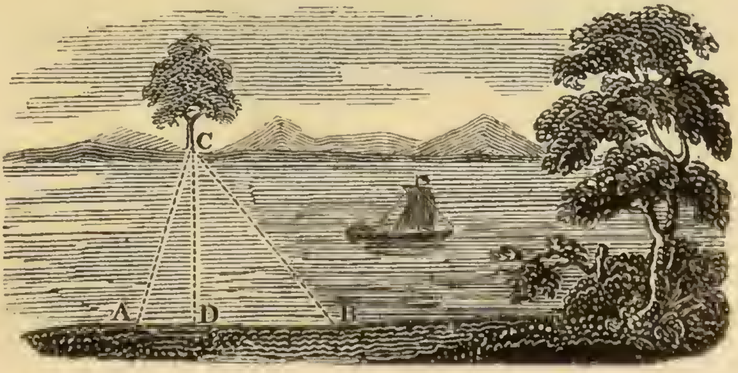
: 593·09 *yards* AC - - - - - - - - - - *Log.* 2·773122

: : Sin. 73° .. 15′ ∠BAC - - - - - - - - Log. 9·981171
: 612·38 *yards* BC - - - - - - - - - - *Log.* 2·787025

Wherefore { 593·09 *yards, distance of* A
612·38 *yards, distance of* B } Ans.

Ex. 20.

Let C be the tree observed, A and B the two stations of observation, and CD the breadth required. It is



(Key to Vol. II. page 24.)

As Sin. 47° .. 48′ ∠ACB - - - Log. ar. co. −10·130296
: Sin. 79° .. 12′ ∠BAC - - - - - - - - Log. 9·992239
: : 500 yards AB - - - - - - - - - - - Log. 2·698970

: 662·99 *yards* BC - - - - - - - - - - *Log.* 2·821505

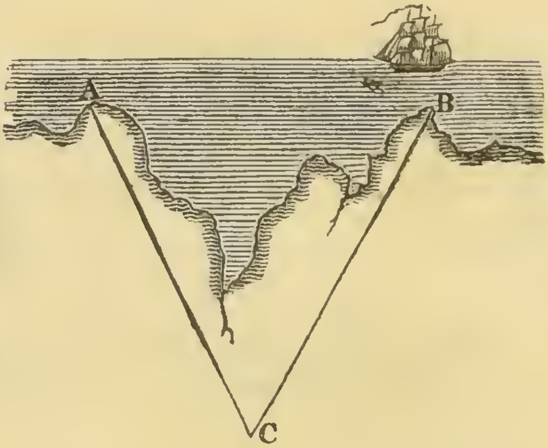
And

As Sin. 90° ∠CDB - - - - - Log. ar. co. −10·———
: Sin. 53° ∠CBD - - - - - - - - - - Log. 9·902349
: : 662·99 yards BC - - - - - - - - - - Log. 2·821505

: 529·48 *yards* CD, *breadth required* - *Log.* 2·723854

Ex. 21.

If A and B represent the two headlands, and C the point inland, there are given AC 735 yards, BC 840 yards, and the ∠C 55° .. 40′; wherefore it will be



As 1575 yards AC+BC - - - - Log. ar. co. −4·802719
: 105 yards BC−AC - - - - - - - - Log. 2·021189
: : Tang. 62° .. 10′ half sup. ∠ACB - - Log. 10·277379

: *Tang.* 7° .. 11′ *half diff.* ∠^{les} A *and* B. *Log.* 9·101287

Hence

The ∠ at A=69° .. 21′ and the ∠ at B=54° .. 59′.

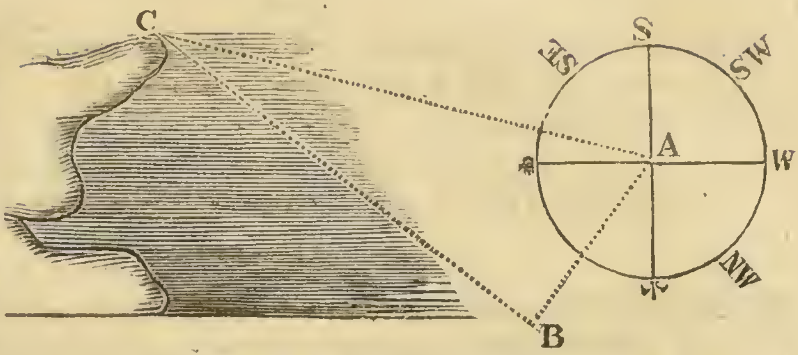
Again

As Sin. 69° .. 21′ ∠A - - - - Log. ar. co. −10·028839
: Sin. 55° .. 40′ ∠C - - - - - - - - - Log. 9·916859
: : 840 yards BC - - - - - - - - - - - Log. 2·924279

: 741·2 *yards* AB, *distance required* - *Log.* 2·869977

(Key to Vol. II. page 24.)

Ex. 22.



Let C be the point of land; A the former and B the latter place of observation. In the triangle ABC are given the angle A 56° .. 15′, the angle B 101° .. 15′, and the side AB 12 miles; consequently the angle C is 22° .. 30′. But

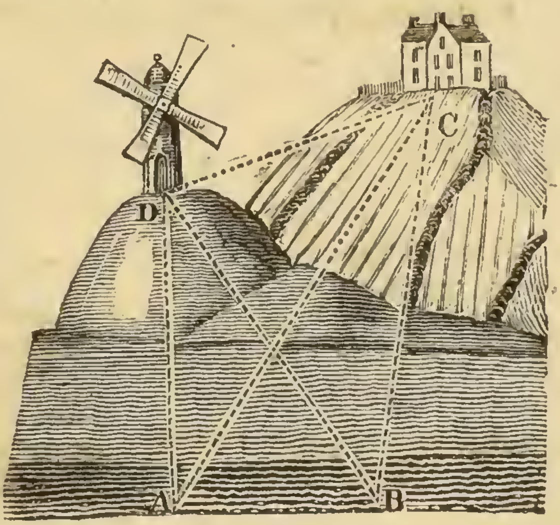
As Sin. 22° .. 30 ∠C - - - - Log. ar. co. −10·417160
: Sin. 56° .. 15 ∠A - - - - - - - - - - Log. 9·919846
: : 12 miles AB - - - - - - - - - - - - - - Log. 1·079181

: 26·0728 *miles* BC, *distance required* *Log.* 1·416187

(Page 25.)

Ex. 23.

If AB in the accompanying diagram be the base line, C the house, and D the mill, there are given the base line AB 600 yards, the ∠DAB 95° 20′, the ∠CAB 58° .. 20, the ∠ABC 98° .. 45′, and the ∠DBA 53° .. 30′; hence the ∠DAC is 37°, the ∠ACB 22° .. 55′, and the ∠ADB 31° .. 10′. But



As Sin. 22° .. 55′ ∠ACB - - - Log. ar. co. −10·409613
: Sin. 98° .. 45′ ∠ABC - - - - - - - - Log. 9·994916
: : 600 yards AB - - - - - - - - - - - - Log. 2·778151

: 1522·92 *yards* AC - - - - - - - - - - - *Log.* 3·182680

(Key to Vol. II. page 25.)

And

As Sin. 31° .. 10′ ∠ADB - -	Log. ar. co.	−10·286065
: Sin. 53° .. 30′ ∠ABD - - - - - - - -	Log.	9·905178
: : 600 yards AB - - - - - - - - - - - -	Log.	2·778151
: 931·96 *yards* AD - - - - - - - - - - - -	*Log.*	2·969394

Also

As 2454·88 yards AC+AD - -	Log. ar. co.	−4·6189697
: 590·96 yards AC−AD - - - - - -	Log.	2·7715581
: : Tang. 71° .. 30′ half sup. ∠DAC -	Log.	10·4754801
: *Tang.* 36° .. 18′ *half the diff. of the* ∠*les* ADC *and* ACD	*Log.*	9·8660079

Wherefore

The ∠ADC=107° .. 48′ and the ∠ACD=35° .. 12′.

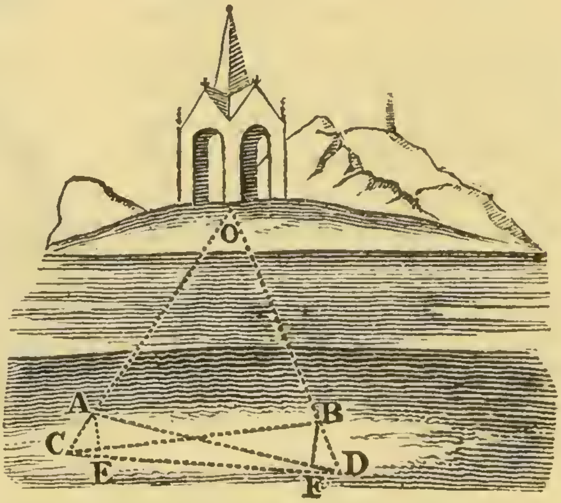
Lastly,

As Sin. 107° .. 48′ ∠ADC - -	Log. ar. co.	−10·0213040
: Sin. 37° .. 0′ ∠DAC - - - - - - -	Log.	9·7794630
: : 1522·92 yards AC - - - - - - - - -	Log.	3·1826771
: 962·59 *yards* DC, *distance required.**	*Log.*	2·9834441

* Differing about 2 yards from the Answer given with the Question.

Ex. 24.

Conceiving the figure constructed as in the marginal diagram, the question gives AB 500 yards; AC and BD each 100 yards; AD 550 yards; and BC 560 yards. Hence it is,



As 560 yds. BC : 600 yds. AB+AC : : 400 yds. AB−AC : 428·57 yds. *the diff. of the segments of* BC *by a perpendicular from* A. And, by ½ *sum* and ½ *diff.* EB=494·28 yards, and EC=65·72 yards. Now

(Key to Vol. II. page 25.)

yds. yds.
As 500 AB : 494·28 EB : : Rad. : *Sin.* 81° .. 20′ ∠EAB.

And

yds. yds.
As 100 AC : 65·72 EC : : Rad. : *Sin.* 41° .. 5′ ∠CAE.

Again,

yds. yds. yds. yds.
As 550 AD : 600 AB+BD : : 400 AB − BD : 436·36 *the diff. of the segments of* AD *by a perpendicular from* B. Wherefore, by $\frac{1}{2}$ *sum* and $\frac{1}{2}$ *diff.* AF=493·18, and FD=56·82. And

yds. yds.
As 100 BD : 56·82 FD : : Rad. : *Sin.* 34° .. 37′ ∠FBD. Also

yds. yds.
As 500 AB : 493·18 AF : : Rad. : Sin. 80° .. 28′ ∠ABF. Consequently the ∠AOB=57° .. 30′, the ∠OAB=57° .. 35′, and the ∠OBA=64° .. 55′.

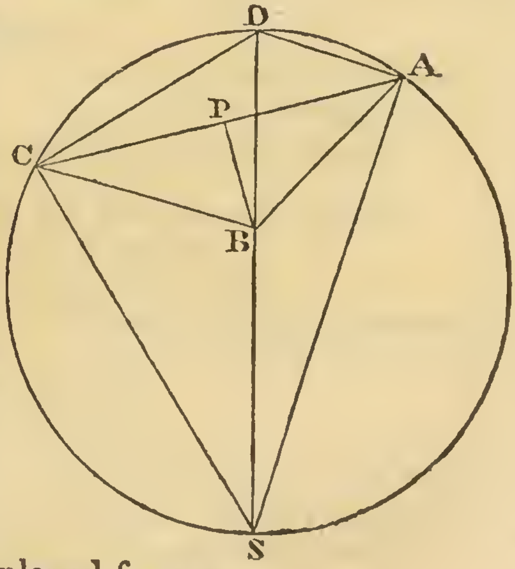
But

yds.
As Sin. 57° .. 30′ ∠AOB : 500 AB.
: : Sin. 57° .. 35′ ∠OAB : 500·47 OB *one of the distances required.*
: : Sin. 64° .. 55′ ∠OBA : 536·81 *nearly,* AO *the other distance sought.* Ans.*

* Differing in the decimal *only* from the Answer given with the Question.

Ex. 25.

1. *Geometrically.*



Construct a triangle, as ABC, having its three sides respectively equal to the three given straight lines, viz. AB $266\frac{1}{4}$ yds. AC 530 yds. and BC $327\frac{1}{2}$ yds. At the point A make the angle CAD equal to the observed angle BSC 29° .. 50′, and at the point C make the angle ACD equal to the observed ASB 13° .. 30′. Then about the triangle ACD describe a circle, as CSAD, *(Eucl.* iv. 5.*)* and join DB. If DB be produced, it will meet the circumference in S, the station of the observer. But, on the scale employed for AB,

(Key to Vol. II. page 25.)

$$\left.\begin{array}{l}\text{SA}\\ \text{SB}\\ \text{SC}\end{array}\right\}\ \text{measures}\ \left\{\begin{array}{ll}757\cdot12 & \text{yds.}\\ 537\cdot & \text{yds.}\\ 655\cdot3 & \text{yds.}\end{array}\right\}\ \text{Ans.}$$

2. *Trigonometrically.*

The figure being as already constructed, from B demit the perpendicular BP upon AC. It is

As Sin. 136° .. 40′ ∠ADC - - Log. ar. co. −10·1635229
: 530 yards AC - - - - - - - - - - - - Log. 2·7242759
: : Sin. 13° .. 30′ ∠ACD - - - - - - - Log. 9·3681853

: 180·29 *yards* AD - - - - - - - - - - *Log.* 2·2559841

: : Sin. 29° .. 50′ ∠DAC - - - - - - - Log. 9·6967745

: 384·21 *yards* DC - - - - - - - - - - *Log.* 2·5845733

Likewise,

As 530 yards AC - - - - - - - Log. ar. co. −3·2757241
: 593·75 yards BC+AB - - - - - - - Log. 2·7736036
: : 61·25 yards BC−AB - - - - - - - Log. 1·7871061

: 68·617 yards *diff. of* AP *and* PC - - *Log.* 1·8364338

Hence, by ½ *sum* and ½ *diff.* AP=230·692 yards, and CP=299·308 yds. Wherefore [*by Theor.* I. *Plane Trigonometry*] the ∠ABP=60° .. 2′, the ∠BAC=29° .. 58′, the ∠CBP=66° .. 3′, and the ∠BCA=23° .. 57′. Also by addition the ∠BCD=71° .. 16′.

Again,

As 711·71 yards CD+CB - - Log. ar. co. −3·1476969
: 56·71 yards CD−CB - - - - - - - Log. 1·7536596
: : Tang. 71° .. 16′ half sup. ∠BCD - Log. 10·4696339

: *Tang.* 13° .. 13′ *half the diff. of the* ∠*les* CDB, CBD. - *Log.* 9·3709904

Consequently the ∠DBC=84° .. 29′, and the ∠CDB=58° .. 3′. But ∠CDA 136° .. 40′− ∠CDB 58° .. 3′= ∠BDA 78° .. 37′.

And

As Sin. 13° .. 30′ ∠ASD	As Sin. 29° .. 50′ ∠CSD
: Sin. 78° .. 37′ ∠SDA	: Sin. 58° .. 3′ ∠CDS
: : 180·29 yards AD	: : 384·21 yards CD
: 757·14 *yards* SA.	: 655·30 *yards* CS.

(Key to Vol. II. page 25.)

Also

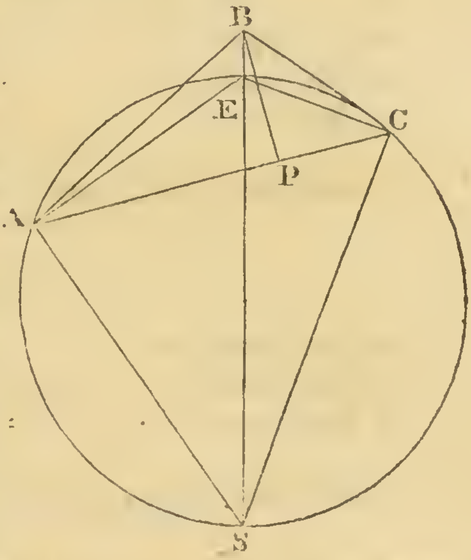
As 266·25 yards AB	As Sin. 138° .. 25′ ∠ ABS.
: 757·14 yards SA	: Sin. 28° .. 5′ ∠ BAS
: : Sin. 13° .. 30′ ∠ BSA	: : 757·14 yards SA
: *Sin.* 138° .. 25′ ∠ ABS.	: 537·10 *yards* SB.

$$\textit{Wherefore}\left\{\begin{array}{l}\text{SA } 757{\cdot}14\\ \text{SB } 537{\cdot}1\\ \text{SC } 655{\cdot}3\end{array}\right\}\textit{yds.}\ \text{Ans.}$$

Ex. 26.

1. *Geometrically.*

Construct a triangle, as ABC, having its three sides respectively equal to the three given straight lines, viz. AB 600 yards, AC 800 yards, and BC 400 yards. At the point A make the angle CAE equal to the observed angle BSC 22° .. 30′, and at the point C make the angle ACE equal to the observed angle ASB 33° .. 45′. Then about the triangle AEC describe *(Eucl.* iv. 5.*)* a circle, as AECS, and join BE. If BE be produced, it will meet the opposite circumference in S, the station of the observer.



But, on the scale employed for AB.

$$\left.\begin{array}{l}\text{SA}\\ \text{SB}\\ \text{SC}\end{array}\right\}\text{measures}\left\{\begin{array}{r}710\frac{1}{3}\text{ yds.}\\ 1042\ \text{ yds.}\\ 934\frac{1}{6}\text{ yds.}\end{array}\right\}\text{Ans.}$$

2. *Trigonometrically.*

The figure being as already constructed, from B demit the perpendicular BP upon AC. The angle ASC=56° .. 15′ the *sum* of the observed angles: and the angle AEC=123° .. 45′ the supplement of the ∠ ASC [*Theor*..54. *Geometry*]. Hence it is

(Key to Vol. II. page 25.)

As Sin. 123° .. 45′ ∠AEC - Log. ar. co. −10·0801536
: 800 yards AC - - - - - - - - - - - - Log. 2·9030900
: : Sin. 22° .. 30′ ∠EAC - - - - - - - Log. 9·5828397

: 368·2 *yards* EC - - - - - - - - - - *Log.* 2·5660833

: : Sin. 33° .. 45′ ∠ECA - - - - - - - Log. 9·7447390

: 534·54 *yards* AE - - - - - - - - - *Log.* 2·7279826

And

yds. yds. yds. yds.
As 800 AC : 1000 AB+BC : : 200 AB−BC : 250 AP−PC.

Therefore AP=525 yds. and PC=275 yds. But

As 275 yards PC - - - - - - Log. ar. co. −3·5606673
: 400 yards BC - - - - - - - - - - - - Log. 2·6020600
: : Rad. CP - - - - - - - - - - - - - - Log. 10· ———

: *Sec.* 46° .. 34′ ∠ACB - - - - - - - - *Log.* 10·1627273

And

As 525 yards AP - - - - - - Log. ar. co. −3·2798407
: 600 yards AB - - - - - - - - - - - - Log. 2·7781513
: : Rad. AP - - - - - - - - - - - - - - Log. 10· ———

: *Sec.* 28° .. 57′ ∠BAC - - - - - - - *Log.* 10·0579920

Consequently the ∠ABC=104° .. 29′, the ∠ECB=12° .. 49′, and the ∠BAE=6° .. 27′.

Again,

As 768·2 yards BC+CE - - - Log. ar. co. −3·1145257
: 31·8 yards BC−CE - - - - - - - Log. 1·5024271
: : Tang. 83° .. 35′½ half sup. ∠BCE - Log. 10·9495616

: *Tang.* 20° .. 14′ *half the diff. of the* ∠^{les} CBE *and* CEB *Log.* 9·5665144

Wherefore the ∠CBS=63° .. 21′½, the ∠ABS=41° .. 7′½, the ∠BCS=94° .. 8′½, and the ∠BAS=105° .. 7′½.

(Key to Vol. II. page 25.)

Now

$$\left.\begin{array}{lllll}
 & (\angle \text{ASB}) & (\angle \text{ABS}) & \text{yds.} & \text{yds.} \\
\text{As} & \text{Sin. } 33^\circ .\ 45' : & \text{Sin. } 41^\circ ..\ 7'\tfrac{1}{2} :: & 600\,\text{AB} : & 710{\cdot}3\ \text{SA} \\
 & (\angle \text{BSC}) & (\angle \text{CBS}) & \text{yds.} & \text{yds.} \\
 & \text{Sin. } 22^\circ .\ 30' : & \text{Sin. } 63^\circ .\ 21'\tfrac{1}{2} :: & 400\,\text{BC} : & 934{\cdot}14\,\text{SC} \\
 & (\angle \text{BSC}) & (\angle \text{BCS}) & \text{yds.} & \text{yds.} \\
 & \text{Sin. } 22^\circ ..\ 30' : & \text{Sin. } 94^\circ ..\ 8'\tfrac{1}{2} :: & 400\,\text{BC} : & 1041{\cdot}85\,\text{SB}
\end{array}\right\}\text{Ans.*}$$

---

* By which it should appear that the Answer given with the Question is correct only in the proximity of the correctness of *Geometrical* Construction to that of *Trigonometrical* Calculation.

MENSURATION OF PLANES.

(Page 27.)

1. *The Area of Parallelograms.*

Ex. 2. $\dfrac{\overset{\text{Chains}}{35{\cdot}25} \times \overset{\text{Chains}}{35{\cdot}25}}{10} = \overset{\text{Acres}}{124{\cdot}25625} = \overset{\text{A.}}{124} .. \overset{\text{R.}}{1} .. \overset{\text{P.}}{1}.$ Ans.

Ex. 3. $\overset{\text{ft.}}{12{\cdot}5} \times \overset{\text{ft.}}{{\cdot}75} = \overset{\text{feet}}{9{\cdot}375} = 9\tfrac{3}{8}$ feet. Ans.

Ex. 4. $\dfrac{\overset{\text{Chains}}{6{\cdot}2} \times \overset{\text{Chains}}{5{\cdot}45}}{10} = \overset{\text{Acres}}{3{\cdot}379} = \overset{\text{A.}}{3} .. \overset{\text{R.}}{1} .. \overset{\text{P.}}{20{\cdot}64}.$ Ans.

Ex. 5. $\dfrac{\overset{\text{ft.}}{37} \times \overset{\text{ft.}}{5\frac{1}{4}}}{9} = 21\tfrac{7}{12}$ square yards. Ans.

(Page 28.)

2. *The Area of Triangles.*

RULE I.

Ex. 2. $\dfrac{\overset{\text{ft.}}{40} \times \overset{\text{ft.}}{30}}{2 \times 9} = 66\tfrac{2}{3}$ square yards. Ans.

(Key to Vol. II. page 28.)

Ex. 3. $$\frac{\overset{\text{ft.}}{24{\cdot}5} \times \overset{\text{ft.}}{25{\cdot}25}}{9} = \overset{\text{sqr. yds.}}{68{\cdot}7361} = 68\tfrac{53}{72} \text{ square yards. Ans.}$$

Ex. 4. $$(\overset{\text{ft.}}{9} \,..\, \overset{\text{in.}}{2}) \times (\overset{\text{ft.}}{11} \,..\, \overset{\text{in.}}{10}) = \overset{\text{sqr. ft.}}{108} \,..\, \overset{\text{pts.}}{5} \,..\, \overset{\text{in.}}{8} = 108\tfrac{17}{36} \text{ sqr. feet. Ans.*}$$

---

* *The Answer being* 108 *ft*: 68 *in. instead of* 108 *ft.* $5\frac{2}{3}$ *in. as given by Hutton, the student is referred to the* 49*th page of the* KEY *for more information.*

(Page 29.)

RULE II.

Ex. 2. $$\frac{\tfrac{1}{2} \times \overset{\text{ft.}}{25} \times \overset{\text{ft.}}{21{\cdot}25} \times {\cdot}7071068^{*}}{9} = \overset{\text{sqr. yds.}}{20{\cdot}86947}.\text{Ans.}$$

Or,

As Rad. - - - - - - - - - - Log. ar. co. $-10\cdot$ ———
: Sin. 45° *the contained* ∠, - - - - Log. 9·8494850
: : 265·625 ft. half prod. of the sides, Log. 2·4242639
: 187·82523 ft. = 20·86947 *sqr. yds. the area sought.* } Log. 2·2737489

* *Nat. Sin.* 45°.

(Page 30.)

RULE III.

Ex. 2. $$\frac{\overset{\text{ft.}}{30} + \overset{\text{ft.}}{40} + \overset{\text{ft.}}{50}}{2} = \overset{\text{ft.}}{60} \text{ half } \textit{Sum} \text{ of the sides. And,}$$

$$\overset{\text{ft.}}{60} - \overset{\text{ft.}}{30} = \overset{\text{ft.}}{30} \text{ First remainder.}$$
$$60 - 40 = 20 \text{ Second remainder.}$$
$$60 - 50 = 10 \text{ Third remainder.}$$

But

$\sqrt{60 \times 30 \times 20 \times 10} = \sqrt{360000} = 600$ square feet, or $66\frac{2}{3}$ square yards. Ans.

(Key to Vol. II. page 30.)

Ex. 3. $\frac{\overset{\text{links}}{2569}+\overset{\text{links}}{4900}+\overset{\text{links}}{5025}}{2}=\overset{\text{links}}{6247}$ half *Sum* of the sides. And

$\overset{\text{links}}{6247}-\overset{\text{links}}{2569}=\overset{\text{links}}{3678}$ First remainder.
$6247-4900=1347$ Second remainder.
$6247-5025=1222$ Third remainder.

But

$$\sqrt{\frac{6247\times3678\times1347\times1222}{100000}}=\overset{\text{AC.}}{61}\,..\,\overset{\text{RO.}}{1}\,..\,\overset{\text{PO.}}{39}.\ \text{Ans.}$$

3. *The Area of Trapezoids.*

Ex. 2. $\frac{\overset{\text{in.}}{15}+\overset{\text{in.}}{11}}{2}\times12\cdot5\text{ feet}=(\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{1})\times(\overset{\text{ft.}}{12}\,..\,\overset{\text{in.}}{6})=13\tfrac{13}{24}$ sq. ft. Ans.

(Page 31.)

Ex. 3. $\frac{\overset{\text{links}}{110}\text{AP}\times\overset{\text{links}}{352}\text{CP}}{2}=19360$ square links, the area of APC.

And

$\frac{(\overset{\text{links}}{352}\text{CP}+\overset{\text{links}}{595}\text{DQ})\times\overset{\text{links}}{635}\text{PQ}}{2}=300672\cdot5$ square links, the area of CPQD.

Again,

$\frac{\overset{\text{links}}{365}\text{BQ}\times\overset{\text{links}}{595}\text{DQ}}{2}=108587\cdot5$ sqr. links, the area of DQB.

But

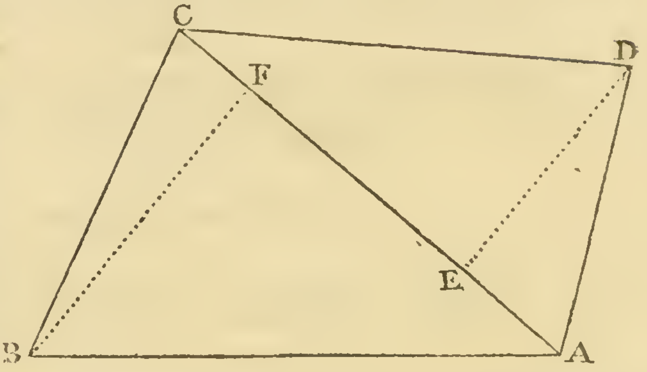
$$\begin{array}{rl}
\overset{\text{sq. links}}{100000})\ \overset{(\text{APC})}{19360}+\overset{(\text{CPQD})}{300672\cdot5}+\overset{(\text{DQB})}{108587\cdot5}\ (& \overset{\text{Acres}}{4\cdot2862}\text{ whole area.}\\
 & 4\\
\hline
\textit{Roods} & 1\cdot1448\\
 & 40\\
\hline
\textit{Perches} & 5\cdot7920
\end{array}$$

(Key to Vol. II. page 31.)

4. *The Area of Trapezia.*

Ex. 2. $\dfrac{(\overset{\text{ft.}}{28}+\overset{\text{ft.}}{33\frac{1}{2}})\times\overset{\text{ft.}}{65}}{2\times 9}=222\frac{1}{12}$ square yards. Ans.

Ex. 3. Construction.



From any convenient scale of equal parts take AC 378 yards, and by the same scale make AE 100 yards, and CF 70 yards. Then from E and F at right angles to AC draw ED, FB; and from the point A with a radius of 220 yards on the scale employed for AC, determine D; also from the point C, with a radius of 265 yards on the same scale, determine B. Lastly, join AB and DC. ABCD is the field required.

Calculation.

$$\sqrt{(\overset{\text{yds.}}{220}\text{AD})^2-(\overset{\text{yds.}}{100}\text{AE})^2}=\overset{\text{yds.}}{196}\ \textit{nearly},\ \text{DE}.$$

And

$$\sqrt{(\overset{\text{yds.}}{265}\text{BC})^2-(\overset{\text{yds.}}{75}\text{CF})^2}=\overset{\text{yds.}}{255\cdot5}\ \text{BF}.$$

$$\text{But}\ \frac{(\overset{\text{yds.}}{196}+\overset{\text{yds.}}{255\cdot5})\times\overset{\text{yds.}}{378}\text{AC}}{2\times4840\ \textit{sq. yds.}}=\overset{\text{A.}}{17}\,..\,\overset{\text{R.}}{2}\,..\,\overset{\text{P.}}{21}.\ \text{Ans.}$$

(Page 32.)

5. *The Area of irregular Polygons.*

EXAMPLE. [See the Figure given with the Problem.]

AC 55 × 9 = 495 area of ABC
AC 55 × 6½ = 357½ - - - - AGC
D*q* 23 × 22 = 506 - - - - GDC
½FD 26 × 12 = 312 - - - - FGD
½FD 26 × 8 = 208 - - - - FDE

Sum 1878½ whole area. Ans.

(Key to Vol. II. page 32.)

Otherwise.

$$\frac{55\,\textsc{ac}\times(18\,\textsc{b}n+13\,\textsc{g}m)}{2}=852\tfrac{1}{2}\ \textsc{abcg}.$$

$$\frac{44\,\textsc{gc}\times 23\,\textsc{d}q}{2}=506\ \textsc{cgd}.$$

$$\frac{52\,\textsc{fd}\times(12\,\textsc{go}+8\,\textsc{ep})}{2}=520\ \textsc{fgde}.$$

Sum $1878\tfrac{1}{2}$ Ans. *as before.*

(Page 33.)

6. *The Area of regular Polygons.*

(Side)(Side) (Tab. area)

Ex. 2. $20\times20\times0{\cdot}4330127=173{\cdot}20508$. Ans.

Ex. 3. $20\times20\times2{\cdot}5980762=1039{\cdot}23048$. Ans.

Ex. 4. $20\times20\times4{\cdot}8284271=1931{\cdot}37084$. Ans.

Ex. 5. $20\times20\times7{\cdot}6942088=3077{\cdot}68352$. Ans.

(Page 36.)

The Length of an Arc of a Circle.

(Const. dec.) (Rad.) feet

Ex. 1. $\cdot01745\times30°\times9\text{ft.}=4{\cdot}7115$. Ans.

(Const. dec.) (Rad.) feet

Ex. 2. $\cdot01745\times12°\tfrac{1}{6}\times10\text{ft.}=2{\cdot}1231$ length of the arc required.

(Page 37.)

The Areas of Circles.

(C.) (D.)

Ex. 2. $\frac{22\times7}{4}=5\tfrac{1}{2}\times7=38\tfrac{1}{2}$. Ans.

Or,

(Circular ar.) (D^2.)

$\cdot7854\times49=38{\cdot}4846$ area required, *nearly as before.*

(Key to Vol. II. page 37.)

Ex. 3. $\overset{(Circul.\ a.)}{\cdot7854} \times \overset{(\text{D}^2.)}{12\frac{1}{4}} = \overset{sq.\ feet}{9\cdot62105} = \overset{sq.\ yds.}{1\cdot069}$. Ans.

Ex. 4. $\overset{(Recip.\ dec.)}{\cdot07958} \times \overset{(\text{C}^2)}{144} = \overset{sq.\ feet.}{11\cdot45952}$. Ans.

(Page 38.)

The Areas of Circular Rings.

Ex. 2. $20 \times 20 = 400$ the square of the greater diameter.
$10 \times 10 = 100$ the square of the less diameter.

300 *difference* of the squares.
Multiply by ·7854

PRODUCT 235·62 area required.

Or,

$20 + 10 = 30$ *sum* of the diameters.
$20 - 10 = 10$ *diff.* of the diameters.

300 rectangle under the *sum* and *diff.**
Multiply by ·7854

PRODUCT 235·62 area required, *as before.*

* EUCLID, Book ii. Prop. 5.

(Page 39.)

The Areas of Sectors of Circles.

Ex. 2. $\dfrac{\overset{(Rad.)}{10} \times \overset{(Arc)}{20}}{2} = 100.$ Ans.

Ex. 3. $\overset{(Rad.)}{25} \times 2 = 50$ the diameter of the circle.

And $50 \times 50 = 2500$ the square of the diameter.

Also $\overset{(Circ.\ a.)}{\cdot7854} \times \overset{(\text{D}^2.)}{2500} = 1963\cdot5$ area of the whole circle.

(Key to Vol. II. page 39.)

But

$$\text{As } \overset{(\textit{Circ.})}{360^\circ} : \overset{(\textit{Giv. arc})}{147^\circ \mathinner{..} 29'} :: \overset{(\textit{Whole area})}{1963{\cdot}5} : \overset{(\textit{Area reqd.})}{804{\cdot}3986}. \quad \text{Ans.}$$

(Page 40.)

The Areas of Segments of Circles.

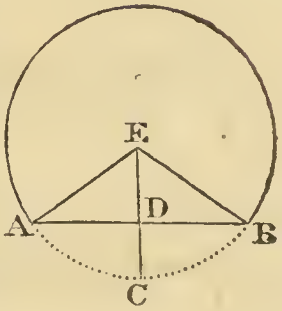
Ex. 3. $\overset{(\text{D.})}{50}) \overset{(\textit{Alt.})}{18} ({\cdot}36$ Quotient.

Entering the *Table of Circular Segments* with ·36, the corresponding area is ·25455. Lastly,

$$\overset{(\textit{Tab. ar.})}{{\cdot}25455} \times \overset{(\text{D}^2.)}{2500} = 636{\cdot}375. \quad \text{Ans.}$$

Ex. 4.

Construction.



From any convenient scale of equal parts take EC 10, and about E at the distance EC describe the circle ABC. Place the right line AB 16 from the same scale, in the circle ABC, [*Eucl.* iv. 1.] and join AE, EB. Bisect AB in D, and through the point D draw EC. The line AB divides the circle into two segments, either of which is the segment whereof the area is required.

Calculation.

As 10 AE - - - - - - - - - - Log. ar. co. —1· ———

: 8 AD - - - - - - - - - - - - - - - - Log. 0·9030900

: : Radius - - - - - - - - - - - - - - Log. 10·-------

: *Sin.* 53° .. 8′ ∠AEC, *nearly* - - - - *Log.* 9·9030900

Wherefore the arc ACB contains 106° .. 16′ *nearly.*

Again,

$$\overset{(\textit{Circ. a.})}{{\cdot}7854} \times \overset{(\text{D}^2.)}{400} = 314{\cdot}16 \text{ the area of the whole circle.}$$

But

$$\text{As } \overset{(\textit{Circ.})}{360^\circ} : \overset{(\textit{Arc ACB})}{106^\circ \mathinner{..} 16'} :: \overset{(\textit{Whole area})}{314{\cdot}16} : \overset{(\textit{Area ACBE})}{92{\cdot}728}.$$

(Key to Vol. II. page 40.)

Now

$\sqrt{100\,\text{AE}^2 - 64\,\text{AD}^2} = \sqrt{36} = 6 = \text{ED}$. And

$\frac{16\,\text{AB} \times 6\,\text{ED}}{2} = \frac{96}{2} = 48$ area of the triangle AEB.

Hence, by subtraction, 44·728 = the area of the segment ABC.

Consequently,

269·432 and 44·728 are the *two* Answers.*

---

* Evincing a defect in the Answer accompanying the Question.

(Page 42.)

The Areas of irregular Figures.

Ex. 2. $\frac{17{\cdot}4 + 24{\cdot}4}{2} = 20{\cdot}9$ mean breadth of the two ends.

And

$\frac{20{\cdot}9 + 20{\cdot}6 + 14{\cdot}2 + 16{\cdot}5 + 20{\cdot}1}{5} \times 84 = 1550{\cdot}64$. Ans.

The Areas of Ellipses.

Ex. 1. $\overset{(\textit{Tra. di.})}{70} \times \overset{(\textit{Conj. di.})}{50} \times \overset{(\textit{Const. dec.})}{\cdot 7854} = 2748{\cdot}9$ area required.

Ex. 2. $\overset{(\textit{Conj. di.})}{24} \times \overset{(\textit{Tra. di.})}{28} \times \overset{(\textit{Const. dec.})}{\cdot 7854} = 339{\cdot}2928$. Ans.

(Page 43.)

The Areas of Elliptic Segments.

Ex. 2. $\overset{(\textit{Alt.})}{10} \div \overset{(\textit{Gr. ax.})}{35} = \cdot 28\tfrac{4}{7}$ the versed sine; to which in the *Table of Circular Segments* corresponds ·18518
Multiply by 875 = 35 × 25, *the axes.*

PRODUCT 162·0325 area required.

(**Key to Vol. II. page 43.**)

Ex. 3. $\overset{(\textit{Alt.})}{5} \div \overset{(\textit{Less ax.})}{25} = \cdot2$ the versed sine; with which, entering the *Table of Circular Segments,* the corresponding area is found ·11182

Multiply by 875 = 35×25, *the two axes.*

Product 97·8425 area sought.

The Areas of Parabolæ and Parabolic Segments.

Ex. 2. $\frac{2}{3}\times\overset{(\textit{Base})}{16}\times\overset{(\textit{Alt.})}{10}=106\frac{2}{3}$. Ans.

MENSURATION OF SOLIDS.

(**Page 44.**)

The Superficies of Prisms and Cylinders.

Ex. 1. $\overset{(\textit{Side})}{20} \times \overset{(\textit{Side})}{20} \times \overset{(\textit{No.})}{6} = 2400$ the surface of the cube.

Ex. 2. $\overset{(\textit{Length})}{20\text{ ft.}} \times \overset{(\textit{Breadth})}{1\cdot5\text{ ft.}} \times \overset{(\textit{Sides})}{3} = \overset{\text{ft.}}{90}$ area of the three sides.

And $\frac{\overset{\text{ft.}}{1\cdot5}+\overset{\text{ft.}}{1\cdot5}+\overset{\text{ft.}}{1\cdot5}}{2}=2\cdot25$ half sum of the three sides of either triangle forming an end of the given prism.

Also $\overset{(\textit{Half sum})}{2\cdot25} - \overset{(\textit{Side})}{1\cdot5} = \cdot75$ one of the three *equal* remainders. But $\sqrt{\cdot75\times\cdot75\times\cdot75\times2\cdot25}=\cdot97432$ area of either end of the prism.

Multiply by 2

Product 1·94864 area of both ends.

Add 90· area of the sides.

Sum 91·94864 whole superficies.

Ex. 3. As $\overset{(\textit{Di.})}{1} : \overset{(\textit{Circumf.})}{3\cdot1416} :: \overset{(\textit{Di.})}{2\text{ ft.}} : \overset{(\textit{Circumf.})}{6\cdot2832\text{ ft.}}$ of the given cylinder.

(Key to Vol. II. page 44.)

And $\overset{\textit{(Circumf.)}}{6{\cdot}2832\text{ ft.}}\times\overset{\textit{(Length)}}{20\text{ ft.}}=\overset{\text{sq. feet.}}{125{\cdot}664}$ convex surface required.

Ex. 4. $(\overset{\text{ft.}}{2{\cdot}5}+\overset{\text{ft.}}{3{\cdot}16\dot{6}}+\overset{\text{ft.}}{2{\cdot}5})\times\overset{\text{ft.}}{2{\cdot}6\dot{6}}=\overset{\text{sq. feet}}{21{\cdot}7\dot{7}}$ the area of the bottom and two ends of the cistern. Also

$(2\times\overset{\text{ft.}}{3{\cdot}16\dot{6}})\times\overset{\text{ft.}}{2{\cdot}5}=\overset{\text{sq. feet}}{15{\cdot}833\dot{3}}$ area of the two sides.

Hence the whole internal area is $37\frac{11}{18}$ square feet.

But $\overset{\text{sq. feet}}{37\frac{11}{18}}\times\overset{\text{lb.}}{7}=\overset{\text{lb.}}{263\frac{5}{18}}$, which at $\overset{d.}{3}$ per lb. = £3 .. 5 .. 9¾ *nearly.*

(Page 45.)

The Superficies of Pyramids and Cones.

Ex. 1. $\dfrac{(\overset{\textit{Perim of the ba.}}{3\text{ ft}+3\text{ ft.}+3\text{ ft.}})\times\overset{\textit{(Sl. ht.)*}}{20\text{ ft.}}}{2}=90$ square feet. Ans.

Ex. 2. As $\overset{\textit{(Di.)}}{1}:\overset{\textit{(Circumf.)}}{3{\cdot}1416}::\overset{\textit{(Di.)}}{8{\cdot}5\text{ ft.}}:\overset{\textit{(Circumf.)}}{26{\cdot}7036\text{ ft.}}$ round the base.

And $\dfrac{\overset{\textit{(Circumf.)}}{26{\cdot}7036\text{ ft.}}\times\overset{\textit{(Sl. ht.)}}{50\text{ ft.}}}{2}=667{\cdot}59$ square feet. Ans.

The Surface of the Frusta of Pyramids and Cones by Sections parallel to the Base.

Ex. 1. $\dfrac{(\overset{\text{ft.}\quad\text{in.}}{3\,..\,4})+(\overset{\text{ft.}\quad\text{in.}}{2\,..\,2})}{2}\times\overset{\textit{(Sides)}}{4}\times\overset{\textit{(Sl. ht.)}}{10\text{ ft.}}=110$ sq. feet. Ans.

Ex. 2. $\dfrac{\overset{\text{ft.}}{6}+\overset{\text{ft.}}{8{\cdot}4}}{2}\times\overset{\textit{(Sl. ht.)}}{12{\cdot}5\text{ ft.}}=90$ square feet. Ans.

(Page 46)

The Solidity of Prisms and Cylinders.

Ex. 1. $\overset{\text{in.}}{24}\times\overset{\text{in.}}{24}\times\overset{\text{in.}}{24}=\overset{\text{cubic in.}}{13824}$. Ans.

---

* The slant height of Pyramids and Pyramidal frusta is an ambiguous expression. Dr. Hutton means the line that in the plane of the perpendicular bisects a side.

(Key to Vol. II. page 46.)

Ex. 2. $\overset{\textit{(Length)}}{(3\text{ ft. }2\text{ in.})} \times \overset{\textit{(Breadth)}}{(2\text{ ft. }8\text{ in.})} \times \overset{\textit{(Thickness)}}{(2\text{ ft. }6\text{ in.})} = 21\tfrac{1}{9}$ solid ft. Ans.

Ex. 3. $\overset{\text{c. in.}}{282}) \, 21\tfrac{1}{9} \times \overset{\text{cub. in.}}{1728} \, (129\tfrac{17}{47}$ gallons of water. Ans.

Ex. 4. Because $3^2+4^2=5^2$ it follows [*Eucl.* i. 48] that the angle contained by the two sides, 3 and 4 respectively, is a right angle. Therefore

$$\frac{3\times4}{2}\times\overset{\textit{(Length)}}{10}\text{ feet}=60\text{ cubic feet, content required.}$$

Ex. 5. $\overset{\text{ft.}}{5{\cdot}5}\times\overset{\text{ft.}}{5{\cdot}5}\times\overset{\textit{(Rec. dec.)}}{{\cdot}07958}\times\overset{\textit{(Length)}}{20\text{ feet}}=\overset{\text{sol. feet.}}{48{\cdot}1459}$. Ans.

(Page 47.)

The Solidity of Pyramids and Cones.

Ex. 1. $\dfrac{\overset{\textit{(Base)}}{30\times30}}{3}\times\overset{\textit{(Alt.)}}{25}=7500$ content required.

Ex. 2. $\dfrac{\overset{\textit{(Sides)}}{3+3+3}}{2}=\dfrac{9}{2}=4{\cdot}5$ half *sum* of the sides.

And $4{\cdot}5-3=1{\cdot}5$ one of the three equal remainders.

But $\sqrt{4{\cdot}5\times1{\cdot}5\times1{\cdot}5\times1{\cdot}5}\times\overset{\textit{(Alt.)}}{30}\div3=3{\cdot}897117\times10$, or 38·97117 the solidity required.

Ex. 3. $\dfrac{\overset{\text{ft.}}{5}+\overset{\text{ft.}}{6}+\overset{\text{ft.}}{7}}{2}=9$ ft. half *sum* of the three sides.

And the three remainders are $\overset{\text{ft.}}{2},\ \overset{\text{ft.}}{3},\ \overset{\text{ft.}}{4}$.

But $\sqrt{9\times2\times3\times4}\text{ ft.}=\sqrt{216}\text{ ft.}=14{\cdot}6969385$ ft. area of the base.

Lastly $\dfrac{\overset{\text{ft.}}{14{\cdot}6969385}\times\overset{\textit{(Alt.)}}{14{\cdot}5\text{ ft.}}}{3}=\overset{\text{solid feet}}{71{\cdot}0352}$. Ans.

(Key to Vol. II. page 47.)

Ex. 4. $\overset{(\textit{Tab. area})}{1{\cdot}7204774} \times \overset{(\textit{Side sqd.})}{4\text{ ft.}} = \overset{\text{sqr. ft.}}{6{\cdot}8819096}$ area of the base.

Multiply by $4\text{ ft} = \frac{1}{3}$ of 12 ft. *altitude.*

PRODUCT 27·5276384 solid feet. Ans.

Ex. 5. $\overset{(\textit{Tab. area})}{2{\cdot}5980762} \times \overset{(\textit{Side sqd.})}{\cdot 25\text{ feet}} = \overset{\text{square feet}}{\cdot 64951905}$ area of the base.

And $\dfrac{\overset{\text{sq. feet}}{\cdot 64951905} \times \overset{(\textit{Alt.})}{6{\cdot}4\text{ ft.}}}{3} = \overset{\text{cubic feet}}{1{\cdot}38564064}$. Ans.

Ex. 6. $\overset{(\textit{Recip. dec.})}{\cdot 07958} \times \overset{(c^2)}{81} = \overset{\text{sq. feet}}{6{\cdot}44598}$ area of the base.

Multiply by $3{\cdot}5\text{ ft.} = \frac{1}{3}$ of $10\frac{1}{2}$ ft. *altitude.*

PRODUCT 22·56093 solid feet. Ans.

(Page 48.)

The Solidity of the Frusta of Cones and Pyramids.

Ex. 1. $\overset{\text{ft.}}{\cdot 5} \times \overset{\text{ft.}}{\cdot 5} = \overset{\text{sq. ft.}}{\cdot 25}$ area of the less end. And
$1{\cdot}25 \times 1{\cdot}25 = 1{\cdot}5625$ - - - - - - greater end.

But $\sqrt{\cdot 25 \times 1{\cdot}5625}\text{ ft.} = \overset{\text{sq. ft.}}{\cdot 625}$ mean proportional.

Again

$$\frac{\overset{\text{sq. ft.}}{\cdot 25} + \overset{\text{sq. ft.}}{\cdot 625} + \overset{\text{sq. ft.}}{1{\cdot}5625}}{3} = \frac{\overset{\text{sq. feet}}{2{\cdot}4375}}{3} = \overset{\text{sq. ft.}}{\cdot 8125}\text{ mean area.}$$

Multiply by 24 *altitude.*

PRODUCT 19·5 solid feet. Ans.

Ex. 2. $\overset{\text{ft.}}{(1{\cdot}5)^2} + \overset{\text{ft.}}{(1{\cdot}5)^2} + (\overset{\text{ft.}}{1{\cdot}5} \times \overset{\text{ft.}}{\cdot 5}) = \overset{\text{sq. ft.}}{2{\cdot}5625}$ sum of the products.

But $\dfrac{\overset{(\textit{Tab. ar.})}{1{\cdot}7204774\text{ ft.}} \times \overset{(\textit{Sum of prods.})}{2{\cdot}5625\text{ ft.}} \times \overset{(\textit{Alt.})}{5\text{ ft.}}}{3} = \overset{\text{cubic feet}}{9{\cdot}31925}$. Ans.

(Key to Vol. II. page 48.)

Ex. 3. $\overset{(\textit{Gr. di}^2.)}{64} + \overset{(\textit{L. di}^2.)}{16} + (\overset{(\textit{Gr. di})}{8} \times \overset{(\textit{L. di.})}{4}) = 112$ sum of the products.

And $\dfrac{\overset{(\textit{Tab. ar.})}{\cdot 7854} \times \overset{(\textit{Sum of pro.})}{112} \times \overset{(\textit{Alt})}{18}}{3} = 527\cdot 7888$. Ans.

Ex. 4. $\overset{(\textit{Gr. di}^2.)}{400} + \overset{(\textit{L. di}^2.)}{100} + (\overset{(\textit{Gr. di})}{20} \times \overset{(\textit{L. di.})}{10}) = 700$ sum of the prods.

And $\dfrac{\overset{(\textit{Tab. ar.})}{\cdot 07958} \times \overset{(\textit{Sum of pro.})}{700} \times \overset{(\textit{Alt.})}{25}}{3} = 464\cdot 216$ solidity required.

Ex. 5. $\overset{(\textit{B. di}^2.)}{784\text{ in.}} + \overset{(\textit{H. di}^2.)}{400\text{ in.}} + (\overset{(\textit{B. di.})}{28\text{ in.}} \times \overset{(\textit{H. di.})}{20\text{ in.}}) = \overset{\text{sq. in.}}{1744}$ sum of the prods.

And $\dfrac{\overset{(\textit{Tab. ar.})}{\cdot 7854\text{ in.}} \times \overset{(\textit{Sum of prods.})}{1744\text{ in.}} \times \overset{(\textit{Alt.})}{40}}{3} = \overset{\text{cub. inches}}{18263\cdot 168}$ the content of the whole Cask. But

$\overset{\text{c. in.}}{231}) \overset{\text{cub. inches}}{18263\cdot 168} (79\cdot 06133$ gallons of wine. Ans.

(Page 50.)

The Superficies of Spheres and Spherical Segments.

Ex. 1. $\overset{(\text{C.})}{22} \times \overset{(\text{D.})}{7} = 154$ the superficies required.

Ex. 2. $\overset{(\text{D}^2.)}{24\text{ in.} \times 24\text{ in.}} \times \overset{(\textit{Const. dec.})}{3\cdot 1416} = \overset{\text{sq. inches.}}{1809\cdot 5616}$ superficies required.

Ex. 3. $\overset{(\textit{Di.})}{7957\cdot 75\text{ mls.}} \times \overset{(\textit{Cir.})}{25000\text{ mls.}} = \overset{\text{square miles}}{198943750}$. Ans.

Ex. 4. As $\overset{(\textit{Di.})}{1} : \overset{(\textit{Cir.})}{3\cdot 1416} :: \overset{(\textit{Di.})}{42\text{ in.}}\ \overset{(\textit{Cir.})}{131\cdot 9472\text{ in.}}$ the circumference of the sphere of which the segment proposed is a part.

But $\overset{(\textit{Whole cir.})}{131\cdot 9472\text{ in.}} \times \overset{(\textit{Alt.})}{9\text{ in.}} = \overset{\text{square inches.}}{1187\cdot 5248}$. Ans.

Ex. 5. As $\overset{(\textit{Di.})}{1} : \overset{(\textit{Cir.})}{3\cdot 1416} :: \overset{(\textit{Di.})}{12\cdot 5\text{ ft.}}\ \overset{(\textit{Cir.})}{39\cdot 27\text{ ft.}}$ the circumference of the sphere of which the zone proposed is a part.

(Key to Vol. II. page 50.)

But $\overset{(\textit{Whole cir.})}{39{\cdot}27\text{ ft.}} \times \overset{(\textit{Alt.})}{2\text{ ft.}} = \overset{\text{sq. ft.}}{78{\cdot}54}$ convex surface required.

(Page 51.)

The Solid Content of Spheres.

Ex. 1. $\overset{(\textit{Di}^3.)}{(12 \times 12 \times 12)} \times \overset{(\textit{Const. dec.})}{{\cdot}5236} = 904{\cdot}7808.$ Ans.

Ex. 2. $\overset{(\textit{Cir.}^3)}{25000^3\text{mls.}} \times \overset{(\textit{Const. dec.})}{{\cdot}01688\text{mls.}} = \overset{\text{square miles}}{263750000000.}$ Ans.

(Page 52.)

The Solidity of Spherical Segments.

Ex. 1. $(\overset{(\textit{Di.})}{8\text{ ft.}} \times 3) - \overset{(\textit{Dbl. alt.})}{4\text{ ft.}} = 20\text{ ft.}$ which $\times^{\text{ed}}$ by $\overset{(\textit{Alt.})^2}{4\text{ ft.}} = \overset{\text{sq. ft.}}{80}$
Multiply by the constant decimal ·5236 feet.

PRODUCT cubic feet 41·888. Ans.

Ex. 2. Here 3 times the square of the radius of the base = 300
The square of the height is 81

Sum 381
Multiply by the height 9

PRODUCT 3429
Multiply by the const. decimal ·5236

Ans. 1795·4244

ARTIFICERS-WORK.

(Page 83.)

BRICKLAYING.

Ex. 1. $\dfrac{\overset{\text{ft. in.}}{(57..3)} \times \overset{\text{ft. in.}}{(24..6)} \times \overset{\textit{h. bricks}}{5}}{3 \textit{ half bricks.}} = \overset{\text{sq. ft. pts. in.}}{2337..8..6} = \overset{\text{rods. yds.}}{8..17\tfrac{2}{3}}.$ Ans.

(Key to Vol. II. page 83.)

Ex. 2. $$\frac{\overset{\text{ft. in.}}{(62\,..\,6)}\times\overset{\text{ft. in.}}{(14\,..\,8)}\times\overset{\textit{h. bricks}}{5}}{9\times 3\ \textit{half bricks}} = 169{\cdot}753 \text{ sq. yds. Ans.}$$

Ex. 3. $$\frac{\overset{(\textit{Length})}{24{\cdot}75\text{ ft.}}\times\overset{(\textit{Height})}{17{\cdot}5\text{ ft.}}\times\overset{\textit{h. bricks}}{4}}{2\times 3\ \textit{half bricks}}\ \overset{\text{sq. ft.}}{\div\ 9} = \overset{\text{sq. yards}}{32{\cdot}083\dot{3}}\text{. Ans.}$$

Ex. 4. $$\frac{\overset{\text{ft. in.}}{(28\,..\,10)}\times\overset{\text{ft.}}{20}\times\overset{\textit{h. br.}}{5}}{3\ \textit{half bricks}} = \overset{\text{sq. ft. 12ths}}{961\,..\,1\tfrac{1}{3}}\text{ the first 20 feet up.}$$

$$\frac{\overset{\text{ft. in.}}{(28\,..\,10)}\times\overset{\text{ft.}}{20}\times\overset{\textit{h. bricks}}{4}}{3\ \textit{half bricks}} = 768\,..\,10\tfrac{2}{3}\text{ the next 20 feet.}$$

$$\overset{\text{ft. in.}}{(28\,..\,10)}\times\overset{\text{ft. in.}}{(15\,..\,8)} = 451\,..\,8\tfrac{2}{3}\text{ brick and half thick.}$$

$$\frac{\overset{\text{ft. in.}}{(28\,..\,10)}\times\overset{\text{ft. in.}}{(10\,..\,6)}{}^{*}\times\overset{\textit{h. bricks.}}{2}}{2\times 3\ \textit{half bricks}} = 100\,..\,11\text{ in the gable.}$$

$$\text{Divide by } 9\,\overset{\text{sq. ft.}}{)\overline{2282\,..\,7\tfrac{2}{3}}(}\,\underline{\overset{\text{sq. yards}}{253{\cdot}626}}\text{. Ans.}$$

* If four courses of bricks make a foot, 42 courses make 10½ feet.

(Page 84.)

MASON-WORK.

Ex. 1. $\overset{\text{ft. in.}}{(53\,..\,6)}\times\overset{\text{ft. in.}}{(12\,..\,3)}\times\overset{\text{ft.}}{2}=\overset{\text{cub. feet.}}{1310\tfrac{3}{4}}$. Ans.

Ex. 2. $\overset{\text{ft. in.}}{(24\,..\,3)}\times\overset{\text{ft. in.}}{(10\,..\,9)}\times\overset{\text{ft.}}{2}=\overset{\text{ft.}}{24{\cdot}25}\times\overset{\text{sq. ft.}}{21{\cdot}5}=\overset{\text{cub. feet.}}{521{\cdot}375}$. Ans.

(Key to Vol. II. page 84.)

Ex. 3. $(\overset{\text{ft.}}{5}\,..\,\overset{\text{in.}}{7})\times(\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{10})=\overset{\text{sq. ft.}}{10}\,..\,\overset{\text{pts.}}{2}\,..\,\overset{\text{in.}}{10}=\overset{\text{sq. ft.}}{10\cdot2361}$

Multiply by 8*s.*

PRODUCT £4 .. 1 .. 10·608. Ans.

Ex. 4. $(\overset{\text{ft.}}{4}\,..\,\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{3}\,..\,\overset{\text{in.}}{2})=\overset{\text{sq. ft.}}{14}\,..\,\overset{\text{pts.}}{3}$

And $(\overset{\text{ft.}}{4}\,..\,\overset{\text{in.}}{4})\times(\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{9})=7\,..\,7$

Sum $21\,..\,10=21\frac{5}{6}$ sq. feet. Ans.

(Page 85.)

CARPENTERS' WORK.

Ex. 1. $\dfrac{(\overset{\text{ft.}}{48}\,..\,\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{24}\,..\,\overset{\text{in.}}{3})}{100\ \textit{sq. feet.}}=\overset{\text{sqrs.}}{11}\,..\,\overset{\text{sq. feet}}{76\frac{1}{8}}$. Ans.

Ex. 2. $\dfrac{(\overset{\text{ft.}}{36}\,..\,\overset{\text{in.}}{3})\times(\overset{\text{ft.}}{16}\,..\,\overset{\text{in.}}{6})}{100\ \textit{sq. feet}}=\overset{\text{sqrs.}}{5}\,..\,\overset{\text{sq. feet}}{98\frac{1}{8}}$. Ans.

Ex. 3. $\dfrac{(\overset{\text{ft.}}{173}\,..\,\overset{\text{in.}}{10})\times(\overset{\text{ft.}}{10}\,..\,\overset{\text{in.}}{7})}{100\ \textit{sq. feet.}}=\overset{\text{squares}}{18\cdot3973}$. Ans.

Ex. 4. $\dfrac{(\overset{\text{ft.}}{52}\,..\,\overset{\text{in.}}{8})\times(\overset{\text{ft.}}{30}\,..\,\overset{\text{in.}}{6})\times3}{2\times100\ \textit{sq. feet}}=\overset{\text{squares}}{24\cdot095}$

Multiply by 10·5 *shillings.*

PRODUCT $£12\,..\,12\,..\,11\,..\,\overset{\text{qr.}}{3\cdot4}$ Ans.

(Key to Vol. II. page 86.)

ft. in. ft. in. sq. ft. pts.
Ex. 5. (83 .. 8) × (12 .. 6) = 1045 .. 10 whole area. Also
ft. in. ft. in. sq. ft. pts.
(7 .. 8) × (3 .. 6) × 3 = 80 .. 3 - - - - - - the three windows.
ft. ft. in.
And 7 × (3 .. 6) = 24 .. 6 - - - - - - the door.

Sum 104 .. 9

Half sum - - - - 52 .. $4\frac{1}{2}$
square yards
Total $1098{\cdot}208\dot{3} = 122{\cdot}02314$

Multiply by 6*sh.*

s. *d.* qrs.
PRODUCT £36 .. 12 .. 1 .. 2·664 Ans.*

* *Dr. Hutton, in his Answer, gives more than three farthings too much.*

SLATING AND TILING.

ft. in. ft. in. sq. ft. pts. in. sq. yds.
Ex. 1. (45 .. 9) × (34 .. 3) = 1566 .. 11 .. 3 = $174\frac{5}{48}$. Ans.

ft. in. ft. in. ft. in.
Ex. 2. (27 .. 5) + 2(1 .. 4) = 30 .. 1 between the ridges at bottom.
ft. in. ft. in. sq. ft. pts. in.
But (43 .. 10) × (30 .. 1) = 1318 .. 7 .. 10
Of which the half is 659 .. 3 .. 11

Sum 1977 .. 11 .. 9 flat and half.

Or, divided by 100 - - - 19·7798 squares, *nearly.*
Multiply by 25·5*sh.*

PRODUCT £25 .. 4 .. $4\frac{1}{2}$. Ans.*

s. *d.*
* *Differing* £0 .. 14 .. $10\frac{3}{4}$ *from Dr. Hutton's Answer.*

OTHERWISE, *and more correctly.*

ft. in. ft. in.
(30 .. 1) × (30 .. 1) = 905·00674 square of the hypothenuse.

(Key to Vol. II. page 86.)

$$\text{And } 2\sqrt{\frac{905{\cdot}00674}{2}} = \overset{\text{ft.}}{42{\cdot}544} \textit{ nearly}, \text{ girt of the ridges.}$$

$$\text{But } \overset{\text{ft.}}{42{\cdot}544} \times \overset{\text{ft.}}{43{\cdot}8\dot{3}} = \overset{\text{sq. feet}}{1864{\cdot}70453} = \overset{\text{squares}}{18{\cdot}6470453}$$

Multiply by 25·5*sh.*

Product £23 .. 15 .. 6. Ans.*

* Differing from Hutton's Answer $£0\,..\,\overset{s.}{13}\,..\,\overset{d.}{11\tfrac{3}{4}}$ in a contrary direction. The mean amount of this and the former result, however, is £24 .. 9 .. $11\tfrac{1}{4}$, being in excess only $5\tfrac{1}{2}$*d.* above the Answer given.

PLASTERING.

(Page 87.)

$$\text{Ex. 1.}\quad \frac{(\overset{\text{ft.}}{43}\,..\,\overset{\text{in.}}{3}) \times (\overset{\text{ft.}}{25}\,..\,\overset{\text{in.}}{6})}{9\ \textit{sq. feet}} = 122\tfrac{1}{2} \text{ square yards. Ans.}$$

$$\text{Ex. 2.}\quad \frac{(\overset{\text{ft.}}{21}\,..\,\overset{\text{in.}}{8}) \times (\overset{\text{ft.}}{14}\,..\,\overset{\text{in.}}{10})}{9\ \textit{sq. feet}} \times \overset{d.}{10} = £1\,..\,\overset{s.}{9}\,..\,\overset{d.}{8\tfrac{3}{4}}. \text{ Ans.}$$

$$\text{Ex. 3.}\quad (\overset{\text{ft.}}{18}\,..\,\overset{\text{in.}}{6}) \times (\overset{\text{ft.}}{12}\,..\,\overset{\text{in.}}{3}) = \overset{\text{sq.ft.}}{226}\,..\,\overset{\text{pts.}}{7}\,..\,\overset{\text{in.}}{6} = 25{\cdot}18 \text{ sq. yds. of ceiling.}$$

$$\text{And } 2(\overset{\text{ft.}}{18}\,..\,\overset{\text{in.}}{6} + \overset{\text{ft.}}{12}\,..\,\overset{\text{in.}}{3}) \times (\overset{\text{ft.}}{10}\,..\,\overset{\text{in.}}{6}) = \overset{\text{sq.ft.}}{645}\,..\,\overset{\text{pts.}}{9} \text{ whole area of wall.}$$

$$\text{Subtr.} \left\{ \begin{array}{l} (\overset{\text{ft.}}{3}\,..\,\overset{\text{in.}}{8}) \times \overset{\text{ft.}}{7} = \overset{\text{sq. ft.}}{25}\,..\,\overset{\text{in.}}{8} \textit{ door.} \\ \overset{\text{ft.}}{5} \times \overset{\text{ft.}}{5} = 25. \textit{fire-place.} \end{array} \right\} 50\,..\,8 \text{ deduction.}$$

$$\textit{Remainder}\quad 595\,..\,1 = \overset{\text{sq. yds.}}{66{\cdot}12} \text{ rendering.}$$

$$\text{But } \overset{\text{sq. yds.}}{66{\cdot}12} \times \overset{d.}{3} + \overset{\text{sq. yds.}}{25{\cdot}18} \times \overset{d.}{8} = £1\,..\,\overset{s.}{13}\,..\,\overset{d.}{3\tfrac{3}{4}}. \text{ Ans.*}$$

* *Being a halfpenny more than the estimate of Dr. Hutton.*

(Key to Vol. II. page 87.)

Ex. 4. $\dfrac{2(\overset{\text{ft.}}{14}..\overset{\text{in.}}{5}+\overset{\text{ft.}}{13}..\overset{\text{in.}}{2})\times(\overset{\text{ft.}}{9}..\overset{\text{in.}}{3})-(\overset{\text{ft.}}{7}\times\overset{\text{ft.}}{4})}{9\ \textit{sq. feet}}=\overset{\text{sq.yds.}}{53}..\overset{\text{ft.}}{5}..\overset{\text{pts.}}{3}..\overset{\text{in.}}{6}$ rendering.

$(\overset{\text{ft.}}{55}..\overset{\text{in.}}{2})\times(\overset{\text{in.}}{8}..\overset{\text{pts.}}{6})=\overset{\text{sq.ft.}}{39}..\overset{\text{pts.}}{0}..\overset{\text{in.}}{11}=4..3..0..11$ of cornice.

And $\dfrac{(\overset{\text{ft.}}{13}..\overset{\text{in.}}{7})\times(\overset{\text{ft.}}{12}..\overset{\text{in.}}{4})}{9\ \textit{sq. feet}}=$ - - - - - - 18..5..6..4 of ceiling.

Whereby the Answer is obvious.

PAINTING.

Ex. 1. $\dfrac{(\overset{\text{ft.}}{65}..\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{12}..\overset{\text{in.}}{4})}{9\ \textit{sq. feet}}=89\tfrac{41}{54}$ square yards. Ans.

Ex. 2. $2(\overset{\text{ft.}}{20}+\overset{\text{ft.}}{14}..\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{10}..\overset{\text{in.}}{4})\ =\ \overset{\text{sq. ft.}}{713}$ whole area of wall.

$$\text{Subtr.}\left\{\begin{array}{l}4(\overset{\text{ft.}}{4}..\overset{\text{in.}}{4})=\overset{\text{sq. ft.}}{17}..\overset{\text{pts.}}{4}\ \textit{fire-place.}\\[6pt] \overset{\text{ft.}}{2}\times\overset{\text{ft.}}{6}\times(\overset{\text{ft.}}{3}..\overset{\text{in.}}{2})=\overset{\text{sq. ft.}}{38}\ \textit{windows.}\end{array}\right\}\ 55..\overset{\text{pts.}}{4}\ \text{deduction.}$$

Remainder $657\tfrac{2}{3}=\overset{\text{sq. yds.}}{73\tfrac{2}{27}}$. Ans.

(Page 88.)

Ex. 3. $2(\overset{\text{ft.}}{24}..\overset{\text{in.}}{6}+\overset{\text{ft.}}{16}..\overset{\text{in.}}{3})\times(\overset{\text{ft.}}{12}..\overset{\text{in.}}{9})=\overset{\text{sq. ft.}}{1039}..\overset{\text{pts.}}{1}..\overset{\text{in.}}{6}$ inside area.

Add

$$\left\{\begin{array}{lll}\text{Outside of the door} & \overset{\text{ft.}}{7}\times(\overset{\text{ft.}}{3}..\overset{\text{in.}}{6})= & \overset{\text{sq.ft.}}{24}..\overset{\text{pts.}}{6}..\overset{\text{in.}}{0}\\[4pt] \text{Inside of the shutters } 2(\overset{\text{ft.}}{7}..\overset{\text{in.}}{9})\times(\overset{\text{ft.}}{3}..\overset{\text{in.}}{6})= & & 54..3..0\\[4pt] \left.\begin{array}{l}\text{Window-breaks, cills, and}\\ \text{soffits } 4(8\text{ft. }6\text{in}+3\text{ft. }6\text{in.})\end{array}\right\} & \times(\overset{\text{ft.}}{1}..\overset{\text{in.}}{3})= & 60..0..0\end{array}\right.$$

Sum 1170..10..6

Deduct fire-place $\overset{\text{ft.}}{5}\times(\overset{\text{ft.}}{5}..\overset{\text{in.}}{6})=$ 27.. 6..0

Remainder $1150..\ 4..6=\overset{\text{sq.yds.}}{127}..\overset{\text{ft.}}{7}..\overset{\text{12ths}}{4\tfrac{1}{2}}$

Multiply by 6*d.*

Product and Answer £3..3..10$\tfrac{3}{4}$

(Key to Vol. II. page 88.)

GLAZING.

Ex. 1. $\overset{\text{ft.}}{4{\cdot}25}\times\overset{\text{ft.}}{2{\cdot}75}=\overset{\text{sq. feet}}{11{\cdot}6875}$. Ans.

Ex. 2. $\dfrac{(\overset{\text{ft.}}{12}\,..\,\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{6}\,..\,\overset{\text{in.}}{9})}{2}\times\overset{d.}{10}=£1\,..\,\overset{s.}{15}\,..\,\overset{d.}{1\tfrac{3}{4}}$. Ans.

Ex. 3. $3(\overset{\text{ft.}}{7}..\overset{\text{in.}}{10}+\overset{\text{ft.}}{6}..\overset{\text{in.}}{8}+\overset{\text{ft.}}{5}..\overset{\text{in.}}{4})\times(\overset{\text{ft.}}{3}..\overset{\text{in.}}{11})\times\overset{d.}{14}=£13..\overset{s.}{11}..\overset{d.}{10\tfrac{1}{2}}$. Ans.

(Page 89.)

Ex. 4. $3(\overset{\text{ft.}}{7}..\overset{\text{in.}}{9}+\overset{\text{ft.}}{6}..\overset{\text{in.}}{6}+\overset{\text{ft.}}{5}..\overset{\text{in.}}{3\tfrac{1}{4}})\times(\overset{\text{ft.}}{3}..\overset{\text{in.}}{9})=\overset{\text{sq. ft.}}{219}..\overset{\text{pts.}}{7}..\overset{\text{in.}}{3\tfrac{3}{4}}$ rectang. frames.

Add $(\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{10\tfrac{1}{2}})\times(\overset{\text{ft.}}{3}\,..\,\overset{\text{in.}}{9})\;=\;7..0..4\tfrac{1}{2}$ oval window.*

Sum $226..7..8\tfrac{1}{4}$ Total.

Multiply by $13d.$

PRODUCT $£12..5..6\tfrac{1}{4}$. Ans.

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* The oval window is computed as rectangular; else the expense is only $£12\,..\,3\,..\,9\tfrac{1}{2}$, the area of the oval being 5·52234375 square feet.

PAVING.

Ex. 1. $\dfrac{(\overset{\text{ft.}}{35}\,..\,\overset{\text{in.}}{4})\times(\overset{\text{ft.}}{8}\,..\,\overset{\text{in.}}{3})}{9\ sq.\ feet}\times(\overset{s.}{3}\,..\,\overset{d.}{4})=£5\,..\,\overset{s.}{7}\,..\,\overset{d.}{11\tfrac{1}{2}}$. Ans.

Ex. 2. $\dfrac{(\overset{\text{ft.}}{27}\,..\,\overset{\text{in.}}{10})\times(\overset{\text{ft.}}{14}\,..\,\overset{\text{in.}}{9})}{9\ sq.\ feet}\times(\overset{s.}{3}\,..\,\overset{d.}{2})=£7\,..\,4\,..\,5\tfrac{1}{4}$. Ans.

Ex. 3. $\overset{\text{ft.}}{45}-(\overset{\text{ft.}}{5}\,..\,\overset{\text{in.}}{3})=\overset{\text{ft.}}{39}\,..\,\overset{\text{in.}}{9}$ breadth laid with pebbles.

(Key to Vol. II. page 89.)

$$\text{And }\frac{\overset{\text{ft.}}{63}\times(\overset{\text{ft.}}{39}\,..\,\overset{\text{in.}}{9})}{9\text{ sq. feet}}\times(\overset{s.}{2}\,..\,\overset{d.}{6})=£34\,..\,\overset{s.}{15}\,..\,\overset{d.}{7\tfrac{1}{2}}\ \textit{pebbled.}$$

$$\text{Also }\frac{\overset{\text{ft.}}{63}\times(\overset{\text{ft.}}{5}\,..\,\overset{\text{in.}}{3})}{9\text{ sq. feet}}\times\overset{s.}{3}\quad=\quad 5\,..\,10\,..\,3\ \textit{foot path.}$$

$$\text{Sum}\quad £40\,..\,5\,..\,10\tfrac{1}{2}.\ \text{Ans.}$$

PLUMBERS' WORK.

$$\text{Ex. 1.}\quad(\overset{\text{ft.}}{30}\,..\,\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{3}\,..\,\overset{\text{in.}}{3})\times\overset{\text{lb.}}{8\tfrac{1}{2}}=\overset{\text{lb.}}{1091}\,..\,\overset{\text{oz.}}{3}.\ \text{Ans.}$$

(Page 90.)

$$\text{Ex. 2.}\quad\overset{\text{ft.}}{43}\times\overset{\text{ft.}}{32}\times\overset{\text{lb.}}{9\cdot831}=\overset{\text{lb.}}{13527\cdot456}\ \text{roofing.}$$

$$\text{And }57\times2\times7\cdot373=\quad 840\cdot522\ \text{guttering.}$$

$$\text{Sum}\quad 14367\cdot978=\overset{\text{cwt.}}{128\cdot286}\ \text{of lead.}$$

$$\text{Multiply by}\qquad 18s.\ \text{per cwt.}$$

$$\text{PRODUCT}\quad £115\,..\,9\,..\,1\tfrac{3}{4}.\ \text{Ans.}$$

TIMBER MEASURING.

The Superficies of Boards.

$$\text{Ex. 1.}\quad(\overset{\text{ft.}}{12}\,..\,\overset{\text{in.}}{6})\times(\overset{\text{ft.}}{0}\,..\,\overset{\text{in.}}{11})\times\overset{d.}{1\tfrac{1}{2}}=£0\,..\,\overset{s.}{1}\,..\,\overset{d.}{5\cdot17}.\ \text{Ans.}$$

By the Sliding Rule.*

Set 12 on B to 11 inches on A; then opposite $12\frac{1}{2}$ feet on B are $11\frac{11}{24}$ feet on A, which multiplied by $1\frac{1}{2}$*d.*, the result is $17\frac{1}{6}$*d.* *nearly.* Ans.

$$\text{Ex. 2.}\quad(\overset{\text{ft.}}{11}\,..\,\overset{\text{in.}}{2})\times(\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{10})=\overset{\text{sq. ft.}}{20}\,..\,\overset{\text{pts.}}{5}\,..\,\overset{\text{in.}}{8}.\ \text{Ans.}$$

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* *The Directions for the* Sliding Rule *being sufficiently intelligible in Hutton's text, it would be superfluous to give more Examples solved in that way.*

(Key to Vol. II. page 90.)

Ex. 3. $(\overset{\text{ft.}}{12}\,..\,\overset{\text{in.}}{9})\times(\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{3})\times\overset{d.}{2\tfrac{1}{2}}=\pounds 0\,..\,\overset{s.}{3}\,..\,\overset{d.}{3\tfrac{3}{4}}$. Ans.

Ex. 4. $(\overset{\text{in.}}{13\tfrac{1}{2}}\times 2)+\overset{\text{in.}}{14\tfrac{1}{2}}+\overset{\text{in.}}{18}+\overset{\text{in.}}{11\tfrac{1}{4}}=\overset{\text{in.}}{70\tfrac{3}{4}}=\overset{\text{ft.}}{5}\,..\,\overset{\text{in.}}{10\tfrac{3}{4}}$. mean breadth.

But $(\overset{\text{ft.}}{5}\,..\,\overset{\text{in.}}{10\tfrac{3}{4}})\times(\overset{\text{ft.}}{17}\,..\,\overset{\text{in.}}{6})\times\overset{d.}{3}=\pounds 1\,..\,\overset{s.}{5}\,..\,\overset{d.}{9\tfrac{1}{2}}$. Ans.

(Page 91.)

The Solidity of Squared Timber.

Ex. 1. $$(\overset{\text{ft.}}{18}\,..\,\overset{\text{in.}}{6})\times\frac{\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{6}+\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{3}}{2}\times\frac{\overset{\text{ft.}}{1}\,..\,\overset{\text{in.}}{3}+\overset{\text{ft.}}{1}}{2}=\overset{\text{c.ft.}}{28}\,..\,\overset{12\text{s.}}{7}\,..\,\overset{144\text{s.}}{4}\,..\,\overset{\text{in.}}{10\tfrac{1}{2}}.\ \text{Ans.*}$$

* Dr. Hutton writes in his Answer 7 inches, meaning (it may be supposed) 1008 inches for $1066\frac{1}{2}$ inches, the exact Answer being 28 *ft.* $1066\frac{1}{2}$ *in.* *See the 49th page of the* KEY.

Ex. 2. $\overset{\text{ft.}}{24{\cdot}5}\times\overset{\text{ft.}}{1{\cdot}04}\times\overset{\text{ft.}}{1{\cdot}04}=\overset{\text{solid feet}}{26{\cdot}4992}$ or $26\tfrac{1}{2}$ cubic ft. *nearly*. Ans.

Ex. 3. $$\frac{\overset{\text{in.}}{19\frac{1}{8}}+\overset{\text{in.}}{9\frac{7}{8}}}{2}=\overset{\text{in.}}{14\tfrac{1}{2}}\ \text{mean breadth and thickness.}$$

But $$\overset{\text{ft.}}{20{\cdot}38}\times\frac{\overset{\text{in.}}{14{\cdot}5}\times\overset{\text{in.}}{14{\cdot}5}}{144\,sq.\,in.}=29{\cdot}7562\ \text{solid feet. Ans.}$$

(Page 92.)

Ex. 4. $$\frac{\overset{\text{ft.}}{178}+\overset{\text{ft.}}{1{\cdot}04}}{2}\times\frac{\overset{\text{ft.}}{1{\cdot}23}+\overset{\text{ft.}}{0{\cdot}91}}{2}\times\overset{\text{ft.}}{27{\cdot}36}=\overset{\text{solid feet.}}{41{\cdot}278}.\ \text{Ans.}$$

The Solidity of Round Timber.

Ex. 1. $\overset{\text{ft.}}{3\tfrac{1}{2}}\times\overset{\text{ft.}}{3\tfrac{1}{2}}=\overset{\text{ft.}}{12\tfrac{1}{4}}$ the square of the *quarter-girt.*
Multiply by $9\tfrac{1}{2}$ ft. the length.

PRODUCT $116\tfrac{3}{8}$ solid feet. Ans.*

---

* Here Dr. Hutton is in error by the Rule (which is fundamentally wrong) 72 cubic inches, or the twenty-fourth part of a solid foot.

Nota Benè. This and the three following Questions being solved in the manner directed, the Answers must not be considered the true content.

(Key to Vol. II. page 92.)

Ex. 2. $\dfrac{\overset{\text{ft.}}{14}+\overset{\text{ft.}}{2}}{2} = \overset{\text{ft.}}{8}$ mean girt; whereof $\frac{1}{4}$ is 2 feet, *quarter-girt.*

But $\overset{\text{ft.}}{2} \times \overset{\text{ft.}}{2} = \overset{\text{ft.}}{4}$ the square of the *quarter-girt.*

Multiply by 24 ft. the length.

PRODUCT 96 solid feet. Ans.

Ex. 3. $\dfrac{\overset{\text{ft.}}{3\cdot5}}{4} = \overset{\text{ft.}}{0\cdot7875}$ the *quarter-girt.* And

$\overset{\text{feet}}{0\cdot7875} \times \overset{\text{feet}}{0\cdot7875} \times \overset{\text{feet}}{14\cdot5} = \overset{\text{solid feet}}{8\cdot992265625}$. Ans.

Ex. 4. $\dfrac{\overset{\text{ft.}}{9\cdot43}+\overset{\text{ft.}}{7\cdot92}+\overset{\text{ft.}}{6\cdot15}+\overset{\text{ft.}}{4\cdot74}+\overset{\text{ft.}}{3\cdot16}}{5} = \overset{\text{ft.}}{6\cdot28}$ mean girt.

Now $\dfrac{\overset{\text{ft.}}{6\cdot28}}{4} = \overset{\text{ft.}}{1\cdot57}$ the *quarter-girt.*

And $\overset{\text{ft.}}{1\cdot57} \times \overset{\text{ft.}}{1\cdot57} = \overset{\text{ft.}}{2\cdot4649}$ square of the *quarter-girt.*
Multiply by 17·25 ft. the length.

PRODUCT 42·519525 solid feet. Ans.

END OF THE MENSURATION.

NATURAL PHILOSOPHY.

(Key to Vol. II. page 151.)

The Laws of Gravity.

Ex. 1. As $1''^2 : 7''^2 :: 16\frac{1}{12}$ ft. : $788\frac{1}{12}$ feet, whole descent.
And $1'' : 7'' :: 32\frac{1}{6}$ ft. : $225\frac{1}{6}$ feet, velocity acquired.

Ex. 2. As $32\frac{1}{6}$ ft. : 100 ft. : : $1'' : 3''\frac{21}{193}$ time of the fall.
And $1''^2 : \overline{3''\frac{21}{193}}|^2 :: 16\frac{1}{12}$ ft. : $155\frac{85}{193}$ ft. passed through.

Ex. 3. As $16\frac{1}{12}$ ft. : 400 ft. : : $1''^2 : \overline{4''\frac{76}{77}}|^2$ time in motion.
And $1'' : 4''\frac{76}{77} :: 32\frac{1}{6}$ ft. : $160\frac{32}{77}$ ft. velocity generated.

QUESTIONS IN PRACTICAL GUNNERY.

(Page 162.)

Ex. 1.

$$\begin{array}{l}
\qquad\quad \text{lb.} \qquad \text{lb.} \quad \text{lb.} \qquad\quad \text{lb.} \qquad\qquad\qquad\quad \text{ft.} \qquad\; \text{ft.} \quad \text{in.}\\
\text{As } \sqrt{196} : \sqrt{18} \because 14 : 4{\cdot}2426407 :: 1600 : 485 - 13 \text{ shell.}\\
\left.\begin{array}{l}\sqrt{90} : \sqrt{8} \because 9{\cdot}486833 : 2{\cdot}828427,\\ \qquad\qquad\qquad\qquad\qquad :: 1600 : 477\end{array}\right\} 10 \text{ shell.}\\
\sqrt{48} : \sqrt{4} \because 6{\cdot}9282032 : 2 :: 1600 : 462 - 8 \text{ shell.}\\
\sqrt{16} : \sqrt{2} \because 4 : 1{\cdot}4142136 :: 1600 : 566 - 5\tfrac{1}{2} \text{ shell.}\\
\sqrt{8} : \sqrt{1} \because 2{\cdot}8284271 : 1 :: 1600 : 566 - 4\tfrac{2}{5} \text{ shell.}
\end{array}$$

Wherefore $\overset{\text{ft.}}{485}$, $\overset{\text{ft.}}{477}$, $\overset{\text{ft.}}{462}$, $\overset{\text{ft.}}{566}$, *and* $\overset{\text{ft.}}{566}$, *respectively*. Ans.

(Key to Vol. II. page 162.)

Ex. 2. As Sin. 2 × (45° ∠ gr. range) Log. ar. co. − 10·
: Sin. 2 × (30° .. 16′ elev. prop.), Log. 9·9398396
: : 1000 yds. given range, - - - - Log. 3· - - - -

: 870·64 *yds, range sought.* - - - *Log.* 2·9398396

(Page 163.)

Ex. 3. As 3750 ft. range found, Log. ar. co. − 4·4259687
: 2810 ft. range proposed, - - - Log. 3·4487063
: : Sin. 2 × (45° trial elevation), - - Log. 10. - - - -

: *Sin.* 2 × $\left\{\begin{matrix}24° .. 16'\\ 65° .. 44'\end{matrix}\right\}$ *elevat. reqd. Log.* 9·8746750

Ex. 4. As Sin. 2(32° .. 12′ giv. elev.) Log. ar. co. − 10·0448741
: Sin. 2 × (45° ∠ of great. range), Log. 10· ———
: : 3250 ft. proposed range, - - - Log. 3·5118834

: 3603·7 *ft. greatest range.* - - - *Log.* 3·5567575

feet
But 3603·7 *ft.* ÷ 2 = 1802 *nearly*, the impetus required.

Again,

$2\sqrt{1802 \times 16\frac{1}{12}}$ ft. $= 2\sqrt{28982\frac{1}{6}}$ ft. = 340·4 ft. the velocity demanded.

And

As 485 ft. tab. velocity, - - - Log. ar. co. − 3·3142583
: 340·4 ft. velocity necessary, - - - Log. 2·5319896
: : √9 lb. tab. charge, - - - - - - - - Log. 0·4771213

: $\sqrt{4·4334 \; lb.}$ = *4lb.* $6\frac{47}{50}$ *oz. charge req.** *Log.* ·3233692

---

* In the Impetus and Velocity we agree nearly with Hutton, but differ more than $\frac{1}{2}$ lb. in the Charge of Powder. Indeed if the Velocity be taken at 340 ft. as he states it, we differ $\frac{3}{4}$ lb. in the Charge.

Ex. 5. As Sin. 2 × (25° .. 12′ actual elevation), Log. ar. co. − 10·1132199
: Sin. 2 × (36° .. 15′ prop. elevation), Log. 9·9794195
: : 3500 ft. range measured, - - - - - Log. 3·5440680

: 4332·1 *ft. range sought.* - - - - - - *Log.* 3·6367074

(Key to Vol. II. page 163.)

Ex. 6. As $\overset{\text{ft.}}{4000} : \overset{\text{ft.}}{3000} \because \overset{\text{ft.}}{4} : \overset{\text{ft.}}{3} :: \overset{\text{lb.}}{9} : \overset{\text{lb.}}{6\frac{3}{4}}$. Ans.

Ex. 7. Because [*Theor.* iii. *page* 156, *vol.* ii.] $T''=2\sqrt{\frac{H}{g}}$ *ft.* and that $g=16\frac{1}{12}$ feet; if, rejecting the fraction, as in comparison small, g be taken=16 feet, it is

$$T''=2\sqrt{\frac{H}{16}}\text{ft.}=2\sqrt{\frac{\frac{1}{4}R}{16}}\text{ft.}=2\times\frac{\sqrt{\frac{1}{4}R}}{\sqrt{16}}\text{ft.}=\frac{1}{2}\sqrt{\frac{1}{4}R},\text{ft.}$$

But $\frac{1}{2}\sqrt{\frac{1}{4}R}$, ft. $=\frac{1}{2}\times\sqrt{\frac{1}{4}}\sqrt{R}$, ft. $=\frac{1}{2}\times\frac{1}{2}\sqrt{R}$, ft. $=\frac{1}{4}\sqrt{R}$, ft.

Wherefore the time *in seconds*$=\frac{1}{4}\sqrt{}$range *in feet.* Ans.

Ex. 8. As Tang. 45° ∠ gr. range, Log. ar. co. −10· ———
: Tang. 32° given ∠ of elevation, Log. 9·7957892
: : 812·5 ft. $\frac{1}{4}$ proposed range, - - Log. 2·9098234

: 507·72 ft. *greatest altitude.* - - - *Log.* 2·7056126

Now $2\sqrt{\frac{507\cdot72}{16\frac{1}{12}}}$ft.$=2\times5''\cdot6=11''\cdot2=11\frac{1}{4}$ seconds *nearly*. Ans.

Ex. 9. 90°−(32° .. 30′)=57° .. 30′. And
(32° .. 30′)−(8° .. 15′)=24° .. 15′. But

The log.-sin. of 57° .. 30′ - - - - is 9·9260292
- - - - - - of 24° .. 15′ - - - - is 9·6135446
The ar. co. of twice the log. cos. 8° .. 15′ is −20·0090356
The log. of 4 (3000 ft. *impetus*) - - - - is 4·0791812

The sum of these 4 *logarithms* - - - - - is 3·6277906 which is the log. of 4244 *feet range* on the ASCENT.

Again

$$\frac{90^\circ-(8^\circ\,..\,15')}{2}=40^\circ\,..\,52'\tfrac{1}{2}.\quad\text{And}$$

The log.-sin. of 57° .. 30′ - - - - is 9·9260292
- - - - - - of 40° .. 52′½ - - - - is 9·8158506
The ar. co. of twice the log.-cos. of 8° .. 15′ is −20·0090356
The log. of 4(3000 ft. *impetus*) - - - - is 4·0791812

The sum of these 4 *logarithms* - - - - - is 3·8300966 which is the log. of 6762·3 *ft. range* on the DESCENT.*

---

* By which it should appear that Dr. Hutton had found 17·3 feet more than the true range *on the descending plane.*

(Key to Vol. II. page 163.)

Ex. 10. Here the impetus (by Art. 98) is 3000 ft. *nearly*. And

$2\sqrt{3000\times16\tfrac{1}{12}}$ ft. $=2\sqrt{48250}$ ft. $=439{\cdot}363$ ft. the Velocity.

But

As $\overset{\text{ft. v.}}{485}$* : $\overset{\text{ft. v.}}{439{\cdot}363}$: : $\sqrt{9}$ lb. : $\sqrt{7{\cdot}38209}$ lb. *nearly*, the Charge.
Ans.†

Ex. 11. Finding the impetus, as in the foregoing Example, to be *nearly* 3000 feet; (for $\sqrt{\frac{2(7{\cdot}375)\text{ lb.}}{196}}\times1600=438{\cdot}4$ feet the velocity: and $\frac{\overline{438{\cdot}4\text{ ft.}}|^2}{2\sqrt{16}}=3003{\cdot}04$ ft. the impetus) it is,

As Rad. : Tang. 8° .. 15′ : : $\frac{3000\text{ ft.}}{2}$: 217·49 ft.

And

As Radius, - - - - - - - - - Log. ar. co. −10· ———
: Cos. 8° .. 15′ ∠ of descent, - - - Log. 9·9954822
: : 6745 ft. given range, - - - - - - Log. 3·8289820

: 6675·2 *ft. horizontal range*. - - - - *Log.* 3·8244642

But

$\frac{6675{\cdot}2}{4}$ ft. − 217·49 ft. = 1451·3 ft.

And

As 1500 ft. ½ the impetus, - Log. ar. co. −4·8239087
: 1451·3 ft. - - - - - - - - - - - - Log. 3·1617572
: : Cos. 8° .. 15′ ∠ of descent, - - - Log. 9·9954822

: *Cos.* 16° .. 46′. - - - - - - - - - - - *Log.* 9·9811481

Now

$$\left.\begin{array}{l}45^\circ-\dfrac{(8^\circ..15')+(16^\circ..46')}{2}=32^\circ..29'\tfrac{1}{2}\\ \text{Also }90^\circ-(32^\circ..29'\tfrac{1}{2}+8^\circ..15')=49^\circ..15'\tfrac{1}{2}\end{array}\right\}\text{Ans.}$$

Otherwise.

Because (*Theor.* i. *page* 159.) $R=\frac{cs}{c^2}\times4a$, or $\frac{c^2R}{4a}=cs$; it is, *(having first found the impetus to charge* $7\frac{3}{8}$ lb. *to be* 3000 *feet* nearly, *as before)* $cs=\frac{\cos.^2\text{ of }(8^\circ..15')\times6745\text{ ft.}}{12000\text{ ft.}}$

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* See Art. 107, page 162, vol. ii.

† This Answer and the Answer given by Dr. Hutton differ in the decimal only.

(Key to Vol. II. page 163.)

In Logarithms.

Twice the log-cosine of 8° .. 15′ = - - 19·9909644
The logarithm of 6745 ft. range = - - 3·8289820
Log. ar. co. of 4(3000 ft. impetus) = - - — 5·9208188

Log. Sum 19·7407652 which

is the logarithm of the cos. of direction above the horizon + the log-sine of direction above the plane. But the angle of direction above the plane is evidently 8° .. 15′ greater than the angle of direction above the horizon. Entering, therefore, the table of log-sines, find the angle of which the log-cosine added to the log-sine of the angle 8° .. 15′ greater, shall approach the nearest to 19·7407652.

Now the log.-cosine of 32° .. 30′ is 9·9260292
Log.-sine of 40° .. 45′ is 9·8147534

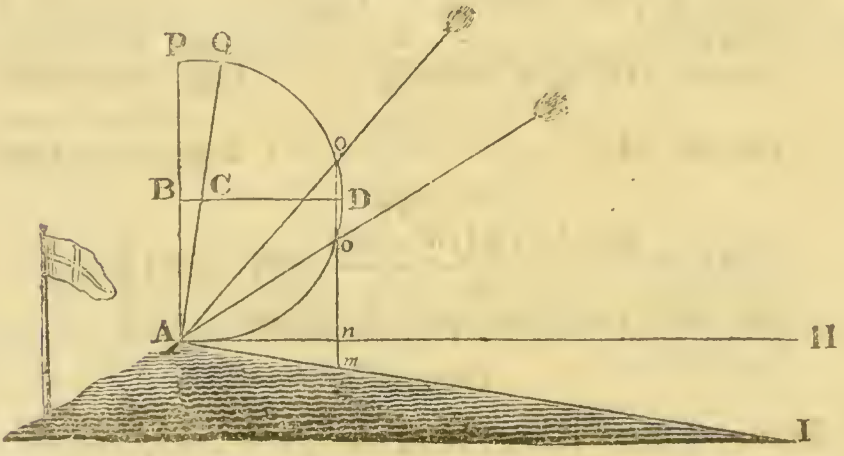
Sum 19·7407826

Also log.-cosine of 49° .. 15′ is 9·8147534
Log.-sine of 57° .. 30′ is 9·9260292

Sum 19·7407826

Consequently 32° .. 30′ *and* 49° .. 15′ *are the elevations required.*

Geometrically.



Draw any right line AH representing the horizon, and at A erect the perpendicular AP = 3000 ft. on any convenient scale of equal parts. Bisect AP in B, and draw BD parallel to AH. Also make the angles PAQ, HAI, each, equal to 8° .. 15′. And from C (the point of intersection of AQ and BD) at the distance CA or CP, describe the arc ADQP. By the scale employed for AP make AI = 6745 ft. and take A*m* one fourth-part of AI. If through the point *m* there be drawn the line *oonm* parallel to AP, the two points *oo* in the circumference ADQP will indicate the elevations required. From A therefore, draw straight lines through the points *o* and *o*; the angle *o*A*n* measured by a line of chords, or goniometer, is $\left\{\begin{matrix}49^\circ .. 15' \\ 32^\circ .. 30'\end{matrix}\right\}$ *nearly*, the two elevations sought.*

* Again the KEY and COURSE are not agreed in the Answer.

Ex. 12. $45^\circ - (8^\circ .. 30') = 36^\circ .. 30'$ the angle of direction above the plane.

And $\sqrt{\frac{2(2304)}{16}}$ ft.† $= 16''\cdot97$ time of greatest horizontal range.

Lastly, As Sin. 45°, - - - - - - - -	Log. ar. co.	−10·1505150
: Sin. 36° .. 30′ - - - - - - - - - - -	Log.	9·7743876
: : 16″·97 time of greatest hor. range,	Log.	1·2296818
: 14″·275 *the time sought.* - - - - -	*Log.*	1·1545844

Otherwise, and more correctly.

Because, [*Theor.* IV. page 159, vol. ii.] $2T = \frac{s}{c}\sqrt{a}$ (if 16 feet, instead of $16\frac{1}{12}$ feet, be substituted for g) it is,

Half the log. of 2304 ft. - - - is	1·6812412	= log. $\sqrt{a}$.
The log. sine of 36° .. 30′ - - is	9·7743876	= log. of s.
The ar. co. log-cos. 8° .. 30′ - is	−10·0047967	= −log. of c.
The log. of 28″·868 *is the* Sum =	1·4604255	= log. of 2T″.

Wherefore 14″·434 *is the time required.*

---

† Because $T = \sqrt{\frac{tR}{g}}$; and that here $t = 1$, $R = 2a$, and $g = 16$ *nearly*.

(Key to Vol. II. page 229.)

HYDROSTATICS.

To find the Specific Gravity of a Solid Body heavier *than Water.*

lb. lb. lb.
EXAMPLE. 10 − 6$\frac{3}{4}$ = 3$\frac{1}{4}$ weight lost in water.

Therefore,

As 3$\frac{1}{4}$ lb. loss in water,
: 10 lb. weight in air,
: : 1000 lb. specific gravity of water,
: 3077 lb. *specific gravity required.*

To find the Specific Gravity of a Solid Body lighter *than Water.*

EXAMPLE. Here the aggregate body weighs 33 lb. in air, and 6 lb. in water. Also, the denser body weighs 18 lb. in air, and 16 lb. in water. But 33 lb. − 6 lb. = 27 lb. lost by the *aggregate* body in water; and 18 lb. − 16 lb. = 2 lb. lost by the *denser* body in water. Therefore,

As 27 lb. − 2 lb. = 25 lb. difference of loss,
: 15 lb. weight of the lighter body in air,
: : 1000 lb. specific gravity of water,
: 600 lb. *specific gravity required.*

(Page 230.)

To find the Specific Gravity of a Fluid.

oz. oz. oz.
EXAMPLE. 40 − 35·61 = 4·39 loss in the fluid. Therefore,

As 40 oz. absolute weight in air,
: 4·39 oz. loss of weight,
: : 7425 oz. specific gravity of cast iron,
: 814·9 oz. *the specific gravity required.*[*]

---

[*] A cubic foot of the fluid in question weighs 185·1 ounces less than Dr. Hutton imagined.

(Key to Vol. II. page 230.)

Specifically to find the Quantity of each of two known Ingredients in a binary Compound.*

EXAMPLE. 8784 − 7320 = 1464 First difference.
9000 − 8784 = 216 Second difference.
9000 − 7320 = 1680 Third difference.

Also 8784 × 1680 = 14757120
9000 × 1464 = 13176000
7320 × 216 = 1581120

$$\left.\begin{array}{l}\text{But, As } 14757120 : 13176000 :: 112\text{lb.} : 100\text{lb. of copper.}\\ \text{And } 14757120 : 1581120 :: 112\text{lb.} : 12\text{lb. of tin.}\end{array}\right\}\text{Ans.}$$

* The Ingredients may be themselves Compound.

(Page 231.)

To determine the Magnitude of Bodies by their Weight.

Ex. 1. As 2520 oz. specific gravity of common stone,
: 1792 oz. absolute weight in ounces,
:: 1728 cubic inches, in a cubic foot,
: $1228\frac{2016}{2520}$ *cubic inches, content required.*

Ex. 2. As 937 oz. specific gravity of gunpowder,
: 16 oz. absolute weight in ounces,
:: 1728 cubic inches, in a cubic foot,
: $29\frac{1}{2}$ *cubic inches, content required.*

(Page 232.)

Ex. 3. As 925 oz. specific gravity of oak,
: 35840 oz. absolute weight in ounces,
:: 1 cubic foot,
: $38\frac{138}{185}$ *cubic feet, content required.*

To determine the Weight of Bodies by their Magnitude.

Ex. 1. $\overset{\text{ft.}}{63} \times \overset{\text{ft.}}{12} \times \overset{\text{ft.}}{12} = 9072$ solid feet in the block.

And, As 1 solid foot,
: 9072 solid feet, content of the block,
:: 2700 oz. specific gravity of marble,
: 24494400 *oz.* = 683·4 *tons, weight required.*

(Key to Vol. II. page 232.)

Ex. 2. As 1728 cubic inches, in a cubic foot,
: 35·25 cubic inches, in 1 pint ale measure,
: : 937 oz. specific gravity of gunpowder,
: 19·1 *oz. the weight required.*

Ex. 3. $\overset{\text{ft.}}{10}\times\overset{\text{ft.}}{3}\times\overset{\text{ft.}}{2\tfrac{1}{2}}=75$ cubic feet in the block.

And, As 1 cubic foot,
: 75 cubic feet, content of the block,
: : 925 oz. specific gravity of dry oak,
: 69375 *oz.* $=4335\frac{15}{16}$ *lb. weight required.*

PNEUMATICS.

(Page 246.)

Ex. 1.
inches
The log. of 29·68 is 1·4724639
The log. of 25·28 is 1·4027771

Difference ·0696868
Multiply by 10592

Product 738·1225856 fathoms.

Again $55^\circ-50^\circ=5^\circ$. *difference* of temperature.

But $\frac{5}{435}\times\overset{\text{fath.}}{738{\cdot}1}=\overset{\text{fath.}}{8{\cdot}5}$ *excess* of 738·1 fathoms.

And $\overset{\text{fath.}}{738{\cdot}1}-\overset{\text{fath.}}{8{\cdot}5}=729{\cdot}6$ fathoms, the height *nearly*.

Ex. 2.
inches
The log. of 29·45 is 1·4690853
The log. of 26·82 is 1·4284588

Difference ·0406265
Multiply by 10000

Product 406·265 fathoms.

(Key to Vol. II. page 246.)

Again 33° − 31° = 2°. *difference* of temperature.

And $\frac{2}{435} \times \overset{\text{fathoms}}{406{\cdot}265} = \overset{\text{fath.}}{1{\cdot}868}$ *deficiency* of $\overset{\text{fath.}}{406{\cdot}2}$ &c.

But $\overset{\text{fath.}}{406{\cdot}265} + \overset{\text{fath.}}{1{\cdot}868} = 408{\cdot}133$ fathoms. Ans.

Or, by the other formula;

The *difference* of the logarithms, *as before*, being ·0406265
Multiply by 10592

Product 430·315888

Again 55° − 33° = 22° *difference* of temperature.

And $\frac{22}{435} \times \overset{\text{fathoms}}{430{\cdot}315} = \overset{\text{fathoms}}{21{\cdot}763}$ *excess* of $\overset{\text{fathoms}}{430{\cdot}3}$ &c.

But $\overset{\text{fathoms}}{430{\cdot}315} - \overset{\text{fathoms}}{21{\cdot}763} = 408{\cdot}552$ fathoms. Ans.*

---

* In this Example the Answer by either formula differs about 11 fathoms from Hutton's Answer.

(Page 247.)

Ex 3. The log. of 1· is - - - ·0000000
The log. of ·25 - - − 1·3979400

Difference ·6020600
Multiply by 10000

Product 6020·6 fathoms. Ans.

(Page 255.)

Ex. 1. 57° − 43° = 14° *diff.* of temperature in the quicksilver.

And $\frac{14}{9600} \times \overset{\text{inches}}{25{\cdot}28} = \overset{\text{inches}}{{\cdot}0368}$ correction for 14° temperature.

Add 25·28 observed alt. of the quicksilver.

Sum 25·3168 Barometer corrected for 57° t.

(Key to Vol. II. page 255.)

Again,

The log. of $\overset{\text{inches}}{29{\cdot}68}$ is 1·4724639
The log. of 25·316 is 1·4033951

Difference ·0690688
Multiply by 10000

Product 690·688 fathoms.

And

$\frac{57^\circ+42^\circ}{2}=49^\circ\tfrac{1}{2}$ mean temperature of the air.

But $49^\circ\tfrac{1}{2}-31^\circ=18^\circ\tfrac{1}{2}$.

Also

$\frac{18{\cdot}5}{435}\times\overset{\text{fathoms}}{690{\cdot}688}=\overset{\text{fathoms}}{29{\cdot}37}$ the deficiency of $\overset{\text{fathoms}}{690{\cdot}6}$ &c.

Therefore $\overset{\text{fathoms}}{690{\cdot}688}+\overset{\text{fathoms}}{29{\cdot}37}=\overset{\text{fathoms}}{720{\cdot}058}$ altitude required.*

* The Altitude here found is about 4 feet more than Dr. Hutton gives.

(Page 256.)

Ex. 2. $41^\circ-38^\circ=3^\circ$ *difference* of temperature by the attached thermometer.

And $\frac{3}{9600}\times\overset{\text{inches}}{29{\cdot}45}=\overset{\text{inches}}{{\cdot}0092}$ correctional altitude for 3° temper.

Add 29·45 observed height of the quicksilver.

Sum 29·4592 Base-alt. corrected for 41° temperature of the quicksilver.

Now the log. of 29·459 is 1·4692180
The log. of 26·82 is 1·4284588

Difference ·0407592
Multiply by 10000

Product 407·592 fathoms.

(Key to Vol. II. page 256.)

Again,

$\frac{31° + 35°}{2}$ = 33° mean temperature by the detached therm.

And

33° − 31° = 2° above the temperature reduced to the formula.

But $\frac{2}{435} \times \overset{\text{fathoms}}{407{\cdot}592} = \overset{\text{fathoms}}{1{\cdot}87}$ the *deficiency* of $\overset{\text{fathoms}}{407{\cdot}5}$ &c.

Therefore the altitude required is 409·462 fathoms.*

---

* Which is *nearly* but not *exactly* the Answer given with the Question.

PRACTICAL QUESTIONS IN MENSURATION.

(Page 259.)

Quest. 1. $\overset{\text{ft.}}{28} \times \overset{\text{ft.}}{20} = \overset{\text{sq. ft.}}{560}$ area of the 1st floor. And
2(14 × 10) = 280 area of the two less floors.

Difference 280 square feet.

Sum 840 square feet = 8·4 SQUARES.

Multiply by 45*s*.

PRODUCT £18 .. 18

Therefore the *Diff.* is 280 sq. feet; and the *Amount* 18 guineas.

Quest. 2. $\overset{\text{sq. yd.}}{1} = \overset{\text{sq. in.}}{1296}$, and $\overset{\text{ft.}}{14} .. \overset{\text{in.}}{3} = 171$ inches in length.

But $\frac{1296}{171} = 7\frac{11}{19}$ inches in width. Ans.

Quest. 3. $\overset{\text{sq. yds.}}{114\frac{2}{3}} = 1032$ square feet.

Now $\frac{\overset{\text{sq. feet}}{1032}}{28\,ft.\,b.} = 36\frac{6}{7}$ feet in length. Ans.

(Key to Vol. II. page 259.)

Quest. 4. $2(\overset{\text{in.}}{7}\times\overset{\text{in.}}{2\frac{1}{2}})=35$ square inches.

But $\dfrac{\overset{\text{sq. in.}}{35}}{3\ \textit{in. t.}}=11\frac{2}{3}$ inches deep. Ans.

Quest. 5. $\overset{\text{sq. yd.}}{1}=\overset{\text{sq. in.}}{1296}$, and

As $\overset{d.}{6}:\overset{d.}{38}::\overset{\text{sq. in.}}{1296}:\overset{\text{sq. in.}}{8208}$ internal area of the cistern.

Now $2(\overset{\text{in.}}{102}\times\overset{\text{in.}}{21})=\overset{\text{sq. in.}}{4284}$ internal area of both sides.

And $\overset{\text{sq. in.}}{8208}-\overset{\text{sq. in.}}{4284}=\overset{\text{sq. in.}}{3924}$ area of the bottom and two ends.

Also $(2\times\overset{\text{in.}}{21})+\overset{\text{in.}}{102}=\overset{\text{in.}}{144}$ length of the bottom, and depth of both ends.

But $\dfrac{\overset{\text{sq. in.}}{3924}}{144\ \textit{in. l.}}=27\frac{1}{4}$ inches wide. Ans.

Quest. 6. $(\overset{\text{ft.}}{47}\,..\,\overset{\text{in.}}{9})\times(\overset{\text{ft.}}{47}\,..\,\overset{\text{in.}}{9})=\overset{\text{sq. ft.}}{2279\frac{1}{16}}$ whole area.
Subtract (47 ft. 9 in.) × 4 ft. = 191 purbeck stone.

Divide by 9 sq. ft. *Rem.* $2088\frac{1}{16}$ paved with flint.

Quotient $232\frac{1}{144}$ square yards.
Multiply by 6*d.* per yard.

Product £5 .. 16 .. 0½. Ans.

Quest. 7. $\sqrt{\left\{(\overset{\text{ft.}}{26\frac{2}{3}}\times\overset{\text{ft.}}{26\frac{2}{3}})-(\overset{\text{ft.}}{22}\times\overset{\text{ft.}}{22})\right\}}=15{\cdot}07$ feet. And
$\sqrt{\left\{(26\frac{2}{3}\times26\frac{2}{3})-(14\times14)\right\}}=22{\cdot}69$ feet.

Sum 37·76 feet *nearly*. Ans.

Quest. 8. 18*d.*=£·075; and,

As £·075 : £100 : : 1 sq. ft. : 1333⅓ sq. ft. area of the court.

Also ½ × 88 ft.=44 feet, half the base.

(Key to Vol. II. page 259.)

But $\dfrac{\overset{\text{sq. feet}}{1333\frac{1}{3}}}{44\text{ ft.}} = 30{\cdot}303$ feet, the perpendicular.

And $\sqrt{44^2+30{\cdot}303^2}\text{ ft.} = \sqrt{1936+918{\cdot}27}\text{ ft.} = 53{\cdot}425$ ft. ½ sum.

Multiply by 2

Product and Answer 106·85 ft.

(Page 260.)

Quest. 9. $\sqrt{86^2-76^2}\text{ ft.} = 40{\cdot}24922$ feet, height of the *lower* column above the Statue. But

$\overset{\text{ft.}}{50} - \overset{\text{feet}}{40{\cdot}24922} = 9{\cdot}75078$ feet, height of the Statue.

Hence the excess of the height of the *higher column* above that of the Statue is 54·24922 feet.

Now $\sqrt{97^2-54{\cdot}24922^2}\text{ ft.} = 80{\cdot}41157$ feet, horizontal distance of the higher column from the center of the figure's base.

And

$\overset{\text{feet}}{76} + \overset{\text{feet}}{80{\cdot}41157} = 156{\cdot}41157$ feet, distance between the Columns.

Again,

$\overset{\text{ft.}}{64} - \overset{\text{ft.}}{50} = \overset{\text{ft.}}{14}$ *difference* of altitude of the Columns.

Wherefore

$\sqrt{156{\cdot}41157^2+14^2}\text{ ft.} = 157{\cdot}03$ feet. Ans.

Quest. 10. $\dfrac{\overset{\text{ft.}}{16{\cdot}5}}{2} = 8{\cdot}25$ feet, circumference of the wheel.

And, As $\overset{\text{(c.)}}{3{\cdot}1416} : \overset{\text{(d.)}}{1} :: \overset{(\textit{Circ.})}{8{\cdot}25\text{ ft.}} : \overset{(\textit{Di.})}{2{\cdot}626\text{ ft.}}$ Ans.

(Key to Vol. II. page 260.)

Quest. 11.* Because the circumferences of circles are to one another as their diameters, and *vice versâ* the diameters in the ratio of the circumferences; it follows (the circumference of the greater circle described being twice the circumference of the less) that the radius of the outer ring is double the radius of the inner. But the distance between the rings is 5 feet, wherefore the diameter of the outer ring is 20 feet.

And, As $\overset{(\text{D.})}{1} : \overset{(\textit{Circ.})}{3{\cdot}14{\cdot}16} :: \overset{(\textit{Di.})}{20\text{ ft.}} : \overset{(\textit{Circ.})}{62{\cdot}832\text{ ft.}}$ Ans.

Quest. 12. Because a guinea for a yard in length is $7s.$ per foot, and that $8d. = \frac{2}{3}$ of a shilling, it is manifest that

As $\frac{2}{3}s. : 7s.$ that is, As $2s. : 21s.$
:: the number of feet *in length* palisaded,
: the number of *square* feet in the area.

Hence, As the number of feet *in length* constituting a side of the Δ,
: the number of *square* feet in the area paved,
$:: \frac{2}{3} : 21 :: 2 : 63.$

But in the table of regular polygons the area of an equilateral triangle is ·4330127 when the side is 1.

Now if x be substituted for the side of the triangle sought, whilst a represents the tabular area ·4330127, it will be

$$\text{As } (a \times x^2) : x : 63 : 2.$$

$$\text{That is, } 2ax^2 = 63x, \text{ or } x = \frac{63}{2a}.$$

Therefore

·8660254)63· - - - - - - (72·746 feet. Ans.

---

* *This Question, like many others in Dr. Hutton's Book, was first taken from a celebrated old Work, but in which the conditions are frequently expressed in rather clumsy, sometimes ungrammatical terms. Although the observation may not exactly apply to the present Example, yet the mention of the height of the wheels was superfluous, inasmuch as (unless it had been meant to be said that the one wheel revolved* only *once and the other twice, a thing impossible) the size of the wheels, if it did not exceed* 10 *feet, had no influence on the Answer; and then only because the revolutions or turns in the Question could not have been performed. It cannot be here objected that the wheels might have been of different heights, since two-wheeled vehicles of this description have never been in use.*

(Key to Vol. II. page 260.)

Quest. 13. $\sqrt{13^2(=\text{AB}^2)+31{\cdot}2^2(=\text{BC}^2)}=33{\cdot}8=\text{AC}$.

Wherefore in the triangle ACD are given the three sides to find the area; which (*page* 29. *vol.* ii.) is 207·322
Add the area of ABC $=\frac{1}{2}\times 13\times 31{\cdot}2$ = 202·8

Sum and Answer 410·122

Quest. 14. $(\overset{\text{ft.}}{24}\,..\,\overset{\text{in.}}{8})\times(\overset{\text{ft.}}{14}\,..\,\overset{\text{in.}}{6})=357\frac{2}{3}$ square feet.
Multiply by 8 lb.

PRODUCT $2861\frac{1}{3}$ lb. of lead.

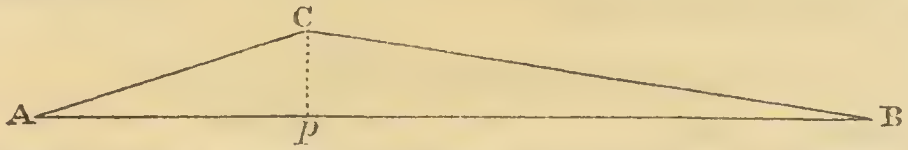
And As $\overset{\text{lb.}}{112} : \overset{\text{lb.}}{2861\frac{1}{3}} :: £{\cdot}9 : £22\,..\,\overset{s.}{19}\,..\,\overset{d.}{10\frac{1}{4}}$. Ans.

Quest. 15.

sq. in. in.	in.			inches left.	
144 ÷ 27 =	5·3	breadth of the	1st cut.	52·6	*length.*
144 ÷ 52·6 =	2·734	- - - - - -	2d cut.	24·266	breadth.
144 ÷ 24·266 =	5·934	- - - - - -	3d cut.	46·732	*length.*
144 ÷ 46·732 =	3·0814	- - - - - -	4th cut.	21·1846	breadth.
144 ÷ 21·1864 =	6·7974	- - - - - -	5th cut.	39·9346	*length.*
144 ÷ 39·9346 =	3·6059	- - - - - -	6th cut.	17·5787	breadth.
144 ÷ 17·5787 =	8·1917	- - - - - -	7th cut.	31·7429	*length.*
144 ÷ 31·7429 =	4·5364	- - - - - -	8th cut.	13·0423	breadth.
144 ÷ 13·0423 =	11·0409	- - - - - -	9th cut.	20·702	*length.*
144 ÷ 20·702 =	6·956	- - - - - -	10th cut.	6·0863	breadth.

There being now less than a square foot remaining, the dimensions are evidently 20·702 inches in *length*, and 6·0863 inches in *breadth*. Ans.

Quest. 16.



Construct any triangle ABC, obtuse angled at C; and let AB be supposed 40 poles; and AC 20 poles. It will be,

As 400 poles half the product of AB × AC,
: 160 poles or 1 acre,
:: Radius,
: *Sin.* 156° .. 25′ *the angle* ACB.

(Key to Vol. II. page 260.)

And

As 60 poles, the sum of AB and AC,
: 20 poles, AB − AC,
: : Tang. 11° .. 47′½ half the supplement of the ∠ACB,
: *Tang.* 3° .. 59′ *the difference of the* ∠^les^ A *and* B.

Hence the angle opposite to the side AC* is 7° .. 48′½, and the angle BAC is 15° .. 46′½.

But As Radius AC,
: Sin. 15° .. 46′½ the ∠BAC,
: : 20 poles, the side AC,
: 5·4344 *poles, the perpendicular cp.*

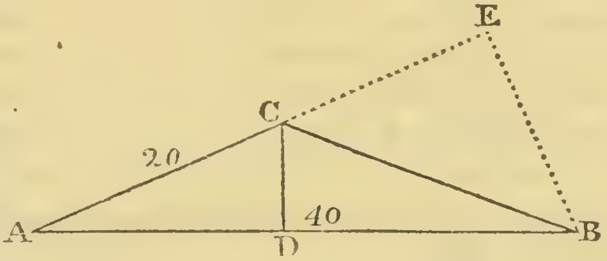
Now $\frac{\overset{\text{poles}}{5{\cdot}4344}}{2} = \overset{\text{poles}}{2{\cdot}7172}$ half the perpendicular *cp.*

And $\frac{\overset{\text{sq. p.}}{160}}{2{\cdot}7172p.} = 58{\cdot}884$ &c. poles AB the third side sought.

* The side BC being by the construction greater than AC.

Or,

If 40 poles be the side opposite to the obtuse angle, then as in the following diagram, AC being 20 poles, it will be $\frac{160\text{ sq. p.}}{20\text{ p.}} \times 2 = 16$ poles BE a perpendicular from B meeting AC produced. Wherefore [*Eucl.* i. 47.] AE=36·66 poles. And by subtraction, CE=16·66 poles; also, by the last-mentioned Theorem, CB=23·099 poles, the *third side of the triangle.*



Otherwise, without producing AC.

$(\overset{\text{(1 acre)}}{160\text{ sq. p.}} \div \overset{\text{(AB)}}{40}\text{ poles}) \times 2 = 8 =$ CD a perpendicular from C upon AB.

And

$AC^2 - CD^2 = AD^2$ Hence $AD = \sqrt{20^2 - 8^2} = \overset{\text{poles}}{\sqrt{336}} = \overset{\text{poles.}}{18{\cdot}3303}$.

But 40 poles AB, − 18·3303 poles AD, = 21·6696 poles DB.

(Key to Vol. II. page 260.)

But $DB^2 + DC^2 = BC^2$. Or in numbers,

$$BC = \sqrt{21{\cdot}6696^2 + 8^2} = 23{\cdot}09 \text{ poles, } \textit{as before.}$$

Consequently 58·884 *poles, and* 23·09 *poles.* Ans.

Courses (*br.*)
Quest. 17. $38 \div 4 = 9\frac{1}{2}$ feet, the height of the triangular gable.

Also $\frac{1}{3} \times 40$ ft. $= 13\frac{1}{3}$ feet, the height of each of the three portions of wall to the eaves.

It is, therefore,

ft. in.	ft. in.	sq.ft. pts.		sq. ft. pts.	
(24 .. 6)×	(13 .. 4)=	326 .. 8	two bricks thick - -	=1306 .. 8	half brick.
(24 .. 6)×	(13 .. 4)=	326 .. 8	brick and half thick	= 980 .. 0	- - - - - -
(24 .. 6)×	(13 .. 4)=	326 .. 8	brick thick - - - - -	= 653 .. 4	- - - - - -
(24 .. 6)×	(4 .. 9)=	116 .. 4½	half brick thick - -	= 116 .. 4½	- - - - -
			Divide by 3)	3056 .. 4½	- - - - -
			Standard thickness	1018·792	of work.

sq. feet sq. feet
But $1018{\cdot}792 \div 272{\cdot}25^* = 3{\cdot}742$ square rods.
Multiply by £5·5 given price per rod.

Product £20 .. 11 .. 7·44. Ans.

* See page 82, vol. ii. of the Course of Mathematics.

(Page 261.)

Quest. 18. 500 feet $=6000$ inches, which $\div$ by $10 = 600$ the number of bricks, end to end, in a half-brick course.

ft. (*bricks*)
But $10 \times 4 \times 600 = 24000$ bricks in the wall, $\frac{1}{2}$ *brick thick.*
Multiply by 3

Product 72000 bricks. Ans.

in. in. in.
Quest. 19. $10 \times 5 \times 3 = 150$ cubic inches in a brick.

(Key to Vol. II. page 261.)

And $\dfrac{\overset{\text{ft.}}{100}\times\overset{\text{ft.}}{100}\times\overset{\text{ft.}}{100}}{3}\overset{(\textit{cub. in. in 1 ft.})}{\times 1728}=576000000$ solid in. of PYRAMID.

But $\overset{\text{cubic inches}}{576000000}\div\overset{\text{cub. in.}}{150}=3840000$ bricks. Ans.

Quest. 20.* $\frac{1}{2}\times\overset{\text{ft.}}{12}\times\overset{\text{ft.}}{16}=96$ sq. feet, area of the triangle.

And, As $\overset{\text{sq. ft.}}{96}:\overset{\text{sq. ft.}}{24}::\overset{\text{ft.}}{12^2}:\overset{\text{ft.}}{6^2}$.

Wherefore the base of the less triangle is 6 feet.†

Again, As $\overset{\text{sq. ft.}}{96}:\overset{\text{sq. ft.}}{24}\because 4:1::\overset{\text{ft.}}{16^2}:\overset{\text{ft.}}{8^2}$.

Consequently the perpendicular sought is 8 feet.

Lastly, $\sqrt{6^2+8^2}$ ft. $=\sqrt{100}$ ft. $=10$ feet the hypothenuse.

* This Question is very *curiously* worded!

† *The base being* 6 *feet or half the base of the greater triangle, it is evident that the perpendicular must be* 8 *feet or half the perpendicular given; and the hypothenuse of the less triangle half the hypothenuse of the greater.*

Quest. 21. $\overset{\text{in.}}{14}\times 2=\overset{\text{in.}}{28}=2\frac{1}{3}$ feet. And

840 links $=554{\cdot}4$ feet. Also
612 links $=403{\cdot}92$ feet. Therefore,

$556{\cdot}7\dot{3}$ feet and $406{\cdot}25\dot{3}$ feet are the diameters of the outer ellipse.

But $\overset{\text{ft.}}{556{\cdot}7\dot{3}}\times\overset{\text{ft.}}{406{\cdot}25\dot{3}}\times{\cdot}7854=177637{\cdot}6\dot{6}$ square feet area.
And $554{\cdot}4\times 403{\cdot}92\times{\cdot}7854=175877{\cdot}17$ square feet area.

Difference $1760{\cdot}49$ *built upon.*

Again $\overset{\text{sq. ft.}}{175877{\cdot}17}\div\overset{\text{sq. ft.}}{43560^*}=\overset{\text{acres}}{4{\cdot}0375}=\overset{\text{A.}}{4}\;..\;\overset{\text{R.}}{0}\;..\;\overset{\text{P.}}{6}$ *inclosed by the walls.*

* The number of square feet in an acre.

Quest. 22. $\overset{\text{in.}}{7}\times\overset{\text{in.}}{7}=\overset{\text{in.}}{49}$ the sqr. of the diameter of the 1st pillar.

And, As $\overset{\text{cub. ft.}}{4}:\overset{\text{cub. ft.}}{40}::\overset{(\text{1st D}^2.)}{49\text{ in.}}:\overset{(\text{2d D}^2.)}{490\text{ inches}}$.

But $\sqrt{\overset{\text{sq. in.}}{490}}=22{\cdot}136$ inches. Ans.

(Key to Vol. II. page 261.)

Quest. 23. $\overset{\text{sq. yds.}}{4840}\div 2=\overset{\text{sq. yds.}}{2420}=\frac{1}{2}$ acre.

And $\overset{\text{sq. yds.}}{2420}\div{\cdot}7854=\overset{\text{sq. yds.}}{3081{\cdot}23}$, square of the diam. of the POND.

Lastly, $\frac{1}{2}\sqrt{\overset{\text{sq. yds.}}{3081{\cdot}23}}=27\frac{3}{4}$ yards, *the radius required.*

Qu. 24. $\overset{\text{feet}}{22{\cdot}75}+(\frac{1}{2}\times\overset{\text{feet}}{22{\cdot}75})+(\overset{\text{ft.}}{1{\cdot}6\dot{6}}$ for the eave boards$)=\overset{\text{feet}}{35{\cdot}791}$.

And $\overset{\textit{(Breadth and half)}}{35{\cdot}791\text{ ft.}}\times\overset{\textit{(Length)}}{32{\cdot}75\text{ ft.}}\div\overset{\text{sq. ft.}}{100}=11{\cdot}7217708$ SQUARES.
Multiply by 15 *shillings.*

PRODUCT £8 .. 15 .. 9½ *nearly.* Ans.

Quest. 25. $\overset{\text{in.}}{9}\times\overset{\text{in.}}{9}=\overset{\text{in.}}{81}$ square of the circumf. of the 1st cable.

Also $12\times 12=144$ square of the circumf. of the 2d cable.

Now, As $\overset{\text{sq. in.}}{81}:\overset{\text{sq. in.}}{144}::\overset{\text{lb.}}{22}:\overset{\text{lb.}}{39\frac{1}{9}}$ weight per yard.
Multiply by 2 yards in a fathom.

PRODUCT $78\frac{2}{9}$ lb. Ans.

Quest. 26.* $(\overset{\text{in.}}{40}\times 2)+\overset{\text{in.}}{26}=\overset{\text{in.}}{106}$ breadth of the bottom and 2 sides spread out. Again,

$\overset{\text{in.}}{106}\times\overset{\text{in.}}{74}=\overset{\text{sq. in.}}{7844}$ area of the bottom and sides.

Also $\overset{\text{in.}}{40}\times(\overset{\text{in.}}{26}\times 2)=2080$ area of both ends.
So $26\times(16\times 3)=1248$ area of the three stays.

Divide the Sum by 144) 11172 ($77{\cdot}58\dot{3}$ square feet in the WHOLE.

Multiply by ·25 cwt. per sq. foot.

PRODUCT 19·39583 cwt. of lead.
Multiply by £1·1 per cwt. for work, &c.
PRODUCT £21 .. 6 .. 8½
Subtract 3 .. 6 balance per Question.
Remainder £21 .. 3 .. 2½ amount of paving.

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* For what purpose is the Question encumbered with the mention of sheet lead $\frac{1}{10}$ of an inch in thickness weighing 5·899 lb. "*the* square foot?"

(Key to Vol. II. page 261.)

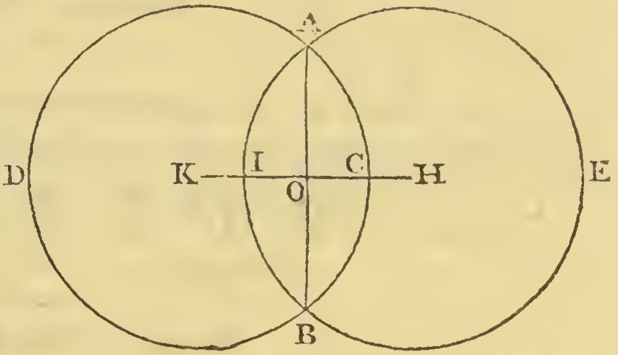
Again,

$$\pounds 21 \,..\, \overset{s.}{3} \,..\, \overset{d.}{2\tfrac{1}{2}} \div \overset{(d.)}{7} = \pounds\frac{21{\cdot}16041}{{\cdot}02916} = 725{\cdot}5$$ a number equal to the number of square feet in the Shop.

$$\text{But, } \overset{\text{sq. ft.}}{725{\cdot}5} \div \overset{\text{ft.}}{22{\cdot}83} = \overset{\text{ft.}}{32} \,..\, \overset{\text{in.}}{0\tfrac{3}{4}}$$ *the length required.*

(Page 262.)

Construction.



Quest. 27. Draw any indefinite right line, and take in it KH corresponding to 30 on some convenient scale of equal parts; then by the same scale make KC and HI, each equal to 25. And about K at the distance KC, and H at the distance HI, describe the circles DAB, BAE, intersecting one another in the points A and B. If the straight line AB be drawn cutting KH in O, it will be, (because KI = CH = 5, and IO = OC = 10)

$$\overset{(Alt.)}{10} \div \overset{(Di.)}{50} = \overset{(Quot.)}{{\cdot}2}$$ *The number wherewith to enter the* Table.

But in the Table of Circular segments the number answering to ·2 is ·111823. Therefore

$$(\overset{(Tab.\ area)}{{\cdot}111823} \times \overset{(Di.)}{50} \times \overset{(Di.)}{50}) \times 2 = 559{\cdot}119.$$ Ans.

Quest. 28.
$$\begin{array}{lllll} \text{in.} & \text{in.} & \text{(ft.)} & & \text{12th pts. of a cub. ft.} \\ 1\frac{1}{4} \times & 1\frac{1}{4} \times & 20 & = & 31{\cdot}25 \\ \frac{7}{8} \times & \frac{7}{8} \times & 50 & = & 38{\cdot}28125 \end{array}$$

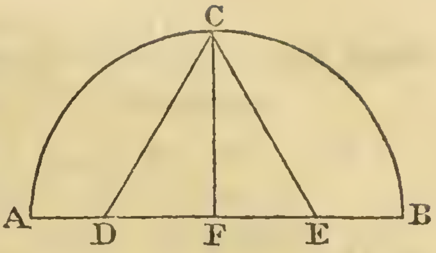
$$\text{Now, As } \overset{\text{12th pts.}}{31{\cdot}25} :: \overset{\text{12th pts.}}{38{\cdot}28125} :: \overset{(\frac{1}{2}\ Ton)}{1120\ \text{lb.}} : \overset{\text{lb.}}{1372} \text{ wt. of the } \overset{\text{ft.}}{50} \text{ proposed.}$$

Multiply by 3½*d.* the price per lb.

Product £20 .. 0 .. 2. Ans.

(Key to Vol. II. page 262.)

Quest 29. Describe an equilateral triangle as DCE, and bisect the base in F. Also produce DE both ways to A and B; and about F at the distance FC describe the semicircle ACB.



Supposing now the area of DCE to be 100, the side of the triangle is $\sqrt{230{\cdot}94}=15{\cdot}17$ &c. because

$$\text{As } \overset{(\textit{Tab. area})}{\cdot 4330127} : \overset{(\textit{Giv. area})}{100} :: \overset{(\textit{Side}^2)}{1} : \overset{(\textit{Side}^2)}{230{\cdot}94}.$$

But $\text{CF}=\text{Radius}=\sqrt{230{\cdot}94-\tfrac{1}{4}(230{\cdot}94)}$ *because* $\text{DF}=\text{FE}=\tfrac{1}{2}\text{DC}$.

Therefore $2\sqrt{230{\cdot}94 \text{ \&c.} - 57{\cdot}73 \text{ \&c.}}=26{\cdot}32148$ &c. Ans.

Quest. 30. 14 lb.=224 oz. And

$$\text{As } \overset{\text{oz.}}{11325} : \overset{\text{oz.}}{224} :: \overset{\text{cub. in.}}{1728} : \overset{\text{cub. in.}}{34{\cdot}17854} \text{ solidity of 1 yard of pipe.}$$

$$\text{Again, } \overset{\text{in.}}{1\tfrac{1}{4}} \times \overset{\text{in.}}{1\tfrac{1}{4}} \times \overset{(\textit{Const. dec.})}{\cdot 7854} \times \overset{\text{in.}}{36} = 44{\cdot}17875 \text{ cavity in 1 yd. of pipe.}$$

$$\text{Now, As } \overset{\text{cub. in.}}{44{\cdot}17875} : \overset{\text{cubic inches}}{(44{\cdot}17875+34{\cdot}17854)} :: \overset{\text{inches}}{(1\tfrac{1}{4}\times 1\tfrac{1}{4})} : \overset{\text{sq. inches}}{2{\cdot}771315}$$

the square of the diameter of the external circumference of the pipe.

$$\text{But } \frac{\sqrt{2{\cdot}771315}-\overset{\text{in.}}{1{\cdot}25}}{2} = \overset{\text{inches}}{\cdot 20736} \text{ \&c. } \textit{thickness required.} \text{ Ans.}$$

$$\text{Quest. 31.} \quad \overset{s.}{2} \,..\, \overset{d.}{4} = \pounds{\cdot}116. \text{ And}$$

$$\text{As } \pounds{\cdot}116 : \pounds 10 :: \overset{\text{sq. ft.}}{1} : \overset{\text{sq. ft.}}{85{\cdot}7143} \text{ the area of the semi-circle.}$$

Multiply by 2

PRODUCT 171·4286 area of the whole circle.

$$\text{But } \overset{\text{sq. feet}}{171{\cdot}4286} \div \overset{(\textit{Const. dec.})}{\cdot 7854} = \overset{\text{sq. feet.}}{218{\cdot}269}, \text{ square of the diameter.}$$

$$\text{Lastly, } \sqrt{\overset{\text{sq. feet}}{218{\cdot}269}} = 14{\cdot}7739 \text{ feet. Ans.}$$

(Key to Vol. II. page 262.)

Quest. 32. As $\overset{\textit{(Di.)}}{1} : \overset{\textit{(Area)}}{\cdot78539816} :: \overset{\textit{(Di.)}}{\tfrac{1}{3}} : \overset{\textit{(Area)}}{\cdot26179939}$.*

But the versed sine of $\cdot26179939$ is $\cdot36753395$.

Now the semichord is a mean proportional between the segments of the diameter; [*Euclid*, iii. 35.]. Therefore

$2\sqrt{(1-\cdot36753395)\times\cdot36753395}=\cdot96426162$ chord to diameter 1.

And

As $\overset{\textit{(Di.)}}{1} : \overset{\textit{(Di.)}}{289} :: \overset{\textit{(Chord)}}{\cdot96426162} : \overset{\textit{(Chord)}}{278\cdot6716}$ *the chord required.*

Quest. 33. $\overset{\text{sq. ft.}}{64\cdot3}=\overset{\text{sq. inches}}{9259\cdot2}$, which $\times^{\text{ed}}$ by $\overset{\text{in.}}{\tfrac{1}{8}}=1157\cdot4$ cubic inches.

And $\overset{\text{cub. in.}}{1157\cdot4}\times\overset{\text{oz.}}{6\tfrac{5}{9}}=\overset{\text{ounces}}{7586\cdot757}=\overset{\text{cwt.}}{4\cdot233}$ of lead in the cistern.

But, As $\overset{\text{cwt.}}{19\tfrac{1}{2}} : \overset{\text{cwt.}}{4\cdot233} :: £21 : £4\ ..\ \overset{s.}{11}\ ..\ \overset{d.}{2\tfrac{1}{4}}$ *nearly.* Ans.

Quest. 34. Because in any sphere the square of the diameter $\times^{\text{ed}}$ by $3\cdot1416$ = the SUPERFICIES, and the cube of the diameter $\times^{\text{ed}}$ by $\cdot5236$ = the SOLIDITY, it follows that the Answer is 6, or $3\cdot1416\div\cdot5236$.

For if x be put for the diameter of the globe, then

$$\begin{array}{r|l}
\text{By the Question.} & x^3\times\cdot5236=x^2\times3\cdot1416 \\
\text{Divide by } x^2. & x\times\cdot5236=1\times3\cdot1416 \\
\text{Divide by } \cdot5236. & x = \dfrac{3\cdot1416}{\cdot5236}=6.\quad \text{Ans.}
\end{array}$$

Quest. 35. $\overset{\text{in.}}{22\tfrac{1}{2}}\times\overset{\text{in.}}{22\tfrac{1}{2}}=\overset{\text{sq. in.}}{506\cdot25}$. And

$(\overset{\text{in.}}{22\tfrac{1}{2}}\times2)\times(\overset{\text{in.}}{22\tfrac{1}{2}}\times2)=\overset{\text{sq. in.}}{2025}$.

But, As $\overset{\text{sq. in.}}{506\cdot25} : \overset{\text{sq. in.}}{2025} :: \overset{\text{bu.}}{3} : \overset{\text{bu.}}{12}$. Ans.

(Page 263.)

Quest. 36. The diameter of the inner circle being 42 inches, and the diameter of the outer circle 56·5 inches, it is

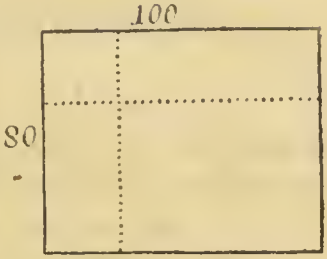
$(\overset{\text{in.}}{56\cdot5^2}-\overset{\text{in.}}{42^2})\times\overset{\textit{(Const. dec.)}}{\cdot7854}=\overset{\text{sq. in.}}{1121\cdot7475}$ area of the curb.

* More briefly $\cdot7854\div3=\cdot2618$ the area, *nearly*, of one third part of a circle whose diameter is Unity.

(Key to Vol. II. page 263.)

Now $\overset{\text{sq. in.}}{1121{\cdot}7475} = 7{\cdot}789$ square feet.
Multiply by $8d.$ per square foot.

PRODUCT £0 .. 5 .. $2\frac{1}{4}$. Ans.



Quest. 37. $\overset{\text{ft.}}{100} \times \overset{\text{ft.}}{80} \div 2 = \overset{\text{sq. ft.}}{4000}$ area of the walk proposed.

And if x be put for the breadth of the walk, it will be

By the Question. | $180x - x^2 = 4000$
Change the signs, complete, &c. | $x = 90 \pm 64{\cdot}0312$
Wherefore | $154{\cdot}0312$ ft. and $25{\cdot}9688$ ft. are the 2 values of x in the equation; but the first being inapplicable, the second or $25{\cdot}9688$ ft. is the Answer.

Quest. 38. $\sqrt{(\overset{\text{ft.}}{26^2} - \overset{\text{ft.}}{10^2})} = \sqrt{(\overset{\text{sq. ft.}}{676} - \overset{\text{sq. ft.}}{100})} = \sqrt{\overset{\text{sq. ft.}}{576}} = \overset{\text{ft.}}{24}$ the part left standing.

And $\overset{\text{ft.}}{24} + \overset{\text{ft.}}{26} = \overset{\text{ft.}}{50}$ *the height required.*

Quest. 39. $\overset{\text{in.}}{60} \times \overset{\text{in.}}{60} \times {\cdot}7854 = \overset{\text{sq. in.}}{2827{\cdot}44}$ area of either circular side of the stone; which, divided by $7 = 403{\cdot}92$ each man's share of the side.

Now $\overset{\text{sq. in.}}{403{\cdot}92} \div {\cdot}7854 = \overset{\text{sq. in.}}{514{\cdot}2856}$ square of the diameter for the last man.

And

$\sqrt{3600 - (514{\cdot}2856 \times 1)} = \overset{\text{inches}}{55{\cdot}5492}$ diameter left by the 1st man.
$\sqrt{3600 - (514{\cdot}2856 \times 2)} = 50{\cdot}7092$ - - - - - - - - - - 2d man.
$\sqrt{3600 - (514{\cdot}2856 \times 3)} = 45{\cdot}3557$ - - - - - - - - - - 3d man.
$\sqrt{3600 - (514{\cdot}2856 \times 4)} = 39{\cdot}2792$ - - - - - - - - - - 4th man.
$\sqrt{3600 - (514{\cdot}2856 \times 5)} = 32{\cdot}0715$ - - - - - - - - - - 5th man.
$\sqrt{3600 - (514{\cdot}2856 \times 6)} = 22{\cdot}6778$ - - - - - - - - - - 6th man.

Wherefore

$$\left.\begin{array}{llll} \overset{\text{inches}}{1st.\ 4{\cdot}4508} - & \overset{\text{inches}}{2nd.\ 4{\cdot}84} - & - \overset{\text{inches}}{3rd.\ 5{\cdot}3535} - & \overset{\text{inches}}{4th\ 6{\cdot}0765} \\ 5th.\ 7{\cdot}2079 - & 6th.\ 9{\cdot}3935 - & 7th.\ 22{\cdot}6778 - & - - - - - - - - \end{array}\right\}\text{Ans.}$$

Quest. 40. $\overset{\text{ft.}}{16\frac{1}{2}}\times\overset{\text{ft.}}{16\frac{1}{2}}=\overset{\text{sq. ft.}}{272\frac{1}{4}}$ area of the *old* Kiln.

And $\sqrt{3\times 272\frac{1}{4}}\text{ sq. ft.}=\sqrt{816\frac{3}{4}}\text{ sq. ft.}=\overset{\text{ft.}}{28}\,..\,\overset{\text{in.}}{7}.$ Ans.

Quest. 41. $\overset{\text{in.}}{3}\times\overset{\text{in.}}{3}\times\overset{\text{in.}}{3}=$ 27 cubic inches in a 3-inch cube.

Also $12\times 12\times 12=1728$ cubic inches in a 12-inch cube.

But $\overset{\text{cub. in.}}{1728}\div\overset{\text{cub. in.}}{27}=64$ cubes. Ans.

Quest. 42. 1 acre=4840 square yards. And

As $355 : 452 :: \overset{\text{sq. yds.}}{4840} : \overset{\text{sq. yds.}}{6162{\cdot}4788}$ square of the diam. sought.

But $\dfrac{\sqrt{\overset{\text{sq. yds.}}{6162{\cdot}4788}}}{2}=39\frac{1}{4}$ yards. Ans.

Quest. 43. As $\overset{\text{(c.)}}{22} : \overset{\text{(d.)}}{7} :: \overset{\text{(Cir.)}}{64\text{ ft.}} : \overset{\text{(Di.)}}{20{\cdot}36\text{ ft.}}$ diameter of the base of the Cone.

Hence the radius of the base is 10·18 ft. *nearly.*

And $\sqrt{118^2+10{\cdot}18^2}$ ft.=118·43 ft. the *slant side* of the cone.

Again $\dfrac{\overset{\text{ft.}}{118{\cdot}43}\times\overset{\text{ft.}}{64}}{2}=\overset{\text{sq. feet}}{3789{\cdot}76}=421{\cdot}08$ square yards.

Multiply by 8*d.* per square yard.

Product £14 .. 0 .. 8·64. Ans.

Quest. 44. $\overset{\text{in.}}{18\frac{1}{2}}\times\overset{\text{in.}}{18\frac{1}{2}}\times\overset{\text{in.}}{8}\times\overset{\text{(Const. dec.)}}{{\cdot}7854}=\overset{\text{cub. in.}}{2150{\cdot}4252}$ in the Bushel.

And $\sqrt{2150{\cdot}425\div 7{\cdot}5\div{\cdot}7854}=19{\cdot}1067$ inches. Ans.

Otherwise.

$\sqrt{\left(\dfrac{\overset{\text{in.}}{18{\cdot}5}\times\overset{\text{in.}}{18{\cdot}5}\times\overset{\text{in.}}{8}}{7{\cdot}5\ in.}\right)}=19{\cdot}1067$ inches *as before.* Ans.

(Key to Vol. II. page 263.)

Quest. 45. $\overset{\text{in.}}{72}\times\overset{\text{in.}}{72}\times\overset{\text{(Const. dec.)}}{3{\cdot}1416}=\overset{\text{sq. inches}}{16286{\cdot}0544}$ surface of the Ball.
Multiply by $3\frac{1}{2}d.$ per sq. inch.

Product £237 .. 10 .. 1. Ans.

Quest. 46. $\dfrac{\overset{\text{ft.}}{4}-\overset{\text{ft.}}{1{\cdot}5}}{2}=\overset{\text{ft.}}{1{\cdot}25}$. And

$(\overset{\text{ft.}}{8}\times\overset{\text{ft.}}{8})-(\overset{\text{ft.}}{1{\cdot}25}\times\overset{\text{ft.}}{1{\cdot}25})=\overset{\text{sq. feet.}}{62{\cdot}4375}$ sqr. of the altitude.

Now $\sqrt{\overset{\text{sq. feet}}{62{\cdot}4375}}=\overset{\text{ft.}}{7{\cdot}9}$ the perpendicular height.

Again,

$(\overset{\text{ft.}}{4}\times\overset{\text{ft.}}{4})+(\overset{\text{ft.}}{1{\cdot}5}\times\overset{\text{ft.}}{1{\cdot}5})+(\overset{\text{ft.}}{4}\times\overset{\text{ft.}}{1{\cdot}5})=\overset{\text{sq. ft.}}{24{\cdot}25}$ sum of the 3 products.
Multiply by ·7854 *constant decimal.*

Product 19·04595 of which one-third is

$\overset{\text{sq. feet}}{6{\cdot}34865}$; this $\times^{\text{ed}}$ by $\overset{\text{ft.}}{7{\cdot}9}=50{\cdot}154335$ solid feet of Marble.
Multiply by 12*s.* per solid foot.

Product £30 .. 1 .. 10¼. Ans.

(Page 264.)

Quest. 47.

As $\sqrt[3]{3}:\sqrt[3]{1}::\overset{\text{in.}}{20}:\overset{\text{in.}}{20\sqrt[3]{\tfrac{1}{3}}}=\overset{\text{in.}}{13{\cdot}867226}$ alt. of the uppermost part.
$\sqrt[3]{3}:\sqrt[3]{2}::20:20\sqrt[3]{\tfrac{2}{3}}=17{\cdot}471606$ - - - do. and middle part.

Difference 3·604380 - - - middle part.

Lastly $\overset{\text{in.}}{20}-\overset{\text{inches}}{17{\cdot}471606}=2{\cdot}528394$ - - - lowest part.

Consequently $\overset{\text{in.}}{2{\cdot}528394}$, $\overset{\text{in.}}{3{\cdot}60438}$,* *and* $\overset{\text{in.}}{13{\cdot}867226}$ *respectively.*
Ans.

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* Differing in the 3d place of decimals from Dr. Hutton.

(Key to Vol. II, page 264.)

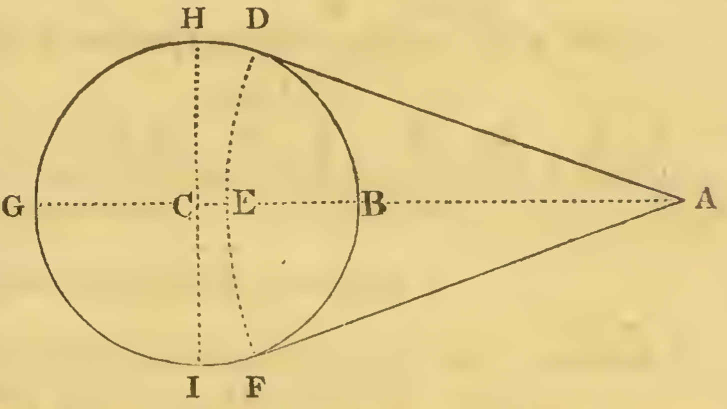
Quest. 48. $\overset{\text{ft.}}{300}\times\overset{\text{ft.}}{200}\times\overset{\text{ft.}}{1}=60000$ cubic feet of earth.

And $2\left\{(\overset{\text{ft.}}{300}\times\overset{\text{ft.}}{8})+(\overset{\text{ft.}}{200}\times\overset{\text{ft.}}{8})\right\}+4(\overset{\text{ft.}}{8}\times\overset{\text{ft.}}{8})\overset{*}{=}\overset{\text{sq. ft.}}{8256}$ area of the surface of the ditch.

But $\overset{\text{cub. feet}}{60000}\div\overset{\text{sq. feet}}{8256}=7\frac{23}{86}$ feet. Ans.

* *The reason will be apparent on constructing the figure.*

Quest. 49.



Let HBIG (in the marginal diagram) represent the earth; and let HI be the true horizon of the observer at A; also F, D two points in the visual horizon, and C the earth's center. Then, because spherical curve surfaces are to one another as their altitudes,† if BE be taken $=\frac{1}{3}$ of BG: it will be

As CE : CB : : CB : CA, that is, As $\frac{2}{3}$ CB : CB : : CB : $\frac{3}{2}$ CB.

Consequently AB=BG, or in other words, the altitude of the observer above the earth's surface is equal to the earth's diameter. Ans.

NOTE.—*Should the Student find any difficulty in tracing the first proportion, he has only to remember that* CB *is supposed equal to* CD, *and that if* D *be connected to* C *and* E *by right lines, the triangles* ACD *and* DCE *are equiangular; hence, As* CE : CD=CB : : CD=CB : CA.

† Cor. ii. page 50. vol. ii.

Quest. 50. $\frac{1}{40}$ in. $\times\frac{1}{40}$ in. $=\frac{1}{1600}$ sq. inch. square of the diameter of the wire.

And $\overset{\text{cub. in.}}{1728}\div\frac{1}{1600}$ sq. in. $\times\overset{(\textit{Const. dec.})}{\cdot785398}=3520252\cdot69329$ inches.

Lastly, $\overset{\text{inches}}{3520252\cdot69}=\overset{\text{yds.}}{97784\cdot797}=\overset{\text{miles}}{55}\,..\,\overset{\text{yds.}}{984\cdot797}$. Ans.

(Key to Vol. II. page 264.)

Quest. 51. The weight of a Cubic inch of cast iron is $\overset{\text{oz.}}{4{\cdot}0509}$.

And As $\overset{\text{oz.}}{4{\cdot}0509} : \overset{\text{oz.}}{(16 \times 24)} :: \overset{\text{in.}}{1} : \overset{\text{in.}}{\frac{128}{1{\cdot}3503}}$ the solidity of the ball.

Hence the diameter of the ball is 5·547 inches.
Add for windage ·1 inch.

Sum 5·647 inches. Ans.

Otherwise.

The diameter of a 24lb. shot, in the Table, is 5·5469 inches.
To which add $\frac{1}{10}$ inch for windage ·1

Sum and Answer 5·6469 in. *as before.*

	lb.	lb.	in.	inches	inches
Quest. 52. As	9 :	1 ::	4^3 :	$1{\cdot}9230^3$ diameter.	1·9622 caliber.
	9 :	2 ::	4^3 :	$2{\cdot}4228^3$ - - - - -	2·4723 - - - -
	9 :	3 ::	4^3 :	$2{\cdot}7734^3$ - - - - -	2·8301 - - - -
	9 :	4 ::	4^3 :	$3{\cdot}0526^3$ - - - - -	3·1149 - - - -
	9 :	6 ::	4^3 :	$3{\cdot}4943^3$ - - - - -	3·5656 - - - -
	9 :	9 ::	4^3 :	$4{\cdot}0000^3$ - - - - -	4·0816 - - - -
	9 :	12 ::	4^3 :	$4{\cdot}4026^3$ - - - - -	4·4924 - - - -
	9 :	18 ::	4^3 :	$5{\cdot}0397^3$ - - - - -	5·1425 - - - -
	9 :	24 ::	4^3 :	$5{\cdot}5469^3$ - - - - -	5·6601 - - - -
	9 :	32 ::	4^3 :	$6{\cdot}1051^3$ - - - - -	6·2297 - - - -
	9 :	36 ::	4^3 :	$6{\cdot}3496^3$ - - - - -	6·4792 - - - -
	9 :	42 ::	4^3 :	$6{\cdot}6844^3$ - - - - -	6·8208 - - - -

Whereby the Answer is obvious.

(Page 265.)

Quest. 53. For a ten-inch Mortar *(the process for the other calibers being the same.)*

Mortar 10 inches, $\begin{cases}\text{windage } \frac{1}{60} \text{ of 10 inches} = \frac{1}{6} \text{ inch.}\\ \text{hollow of the shell } \frac{7}{10} \text{ of 10 in.} = 7 \text{ inches.}\end{cases}$

And 10 in. $- \frac{1}{6}$ inch $= 9\frac{5}{6}$ inches, *external diameter of the Shell.*

(Key to Vol. II. page 265.)

Now { cube of the external diameter is 950·828 inches.
cube of the internal diameter is 343·000 inches.

Difference 607·828 inches.

But $\frac{9}{64}$[*] × 607·828 inches = 85·476 lb. *weight of the shell.*

(*Int. di*^{3}.) cub. in.[†]
And 343 ÷ 57·3 = 5·986 lb. *of gunpowder requisite.*

Proceeding in the same manner with the other given calibers, the following results are obtained.

Caliber	Diameter	Weight	Powder
in. 4·6	*in.* 4·523	*lb.* 8·320	*lb.* 0·583
5·8	5·703	16·677	1·168
8·	7·867	43·764	3·065
13·	12·783	187·791	13·151.

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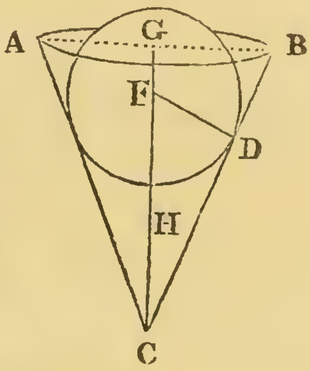
[*] See page 270, vol. ii. of the COURSE.
[†] Page 271 same volume.

Quest. 54. Let ABC represent the conical glass, F the center of the sphere, and FD a radius at right angles to BC the slant side of the cone; if a circle be described about F at the distance FD it will represent the globe *in plano*: H, therefore, is the lowest point of the sphere, and CHF are in the same straight line, which, if produced to G, will bisect AB.



By the Question AG = 2·5 inches, and GC = 6 inches.

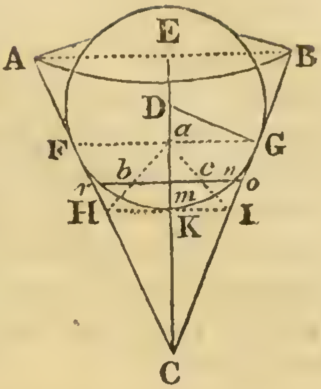
But $\sqrt{AG^2+GC^2}=AC$; that is, $AC=\sqrt{42{\cdot}25}=6{\cdot}5$ inches.

Now, *(because of similar triangles)* it is,
in. in. in. in.
As 2·5 [AG] : 6·5 [AC] : : 2 [FD] : 5·2 [FC].
Consequently FG = ·8 inch, and GH = 2·8 inches.

Again, (*page* 51. *vol.* ii. *Prob.* ix.) the solidity of the segment
in. in. in. in. (*Const. dec.*)
immersed is $\{(4\times3)-(2{\cdot}8\times2)\}\times2{\cdot}8\times2{\cdot}8\times{\cdot}5236=26{\cdot}272$ cubic inches.

cub. in. cub. in. pts.
Lastly, As 282 : 26·272 : : 8 : $\frac{3}{4}$ of a pint *nearly*. Ans.

(Key to Vol. II. page 265.)



Quest. 55. Let *ro* be the surface of the water round the ball, the cone being perfectly upright; and let HKI be a horizontal tangent at K to the sphere immersed; F*a*G a line parallel to HI from G a point of contact of the globe and cone, D the center of the sphere, and DG a radius at right angles to BC the slant side of the glass. Join AB, *a*H, *a*I; and from C draw CE perpendicular to AB; CE passes through the points K, D, and bisects AB in E.

By the Question AB$=$5 inches, EC$=$6 inches, and DG or DK$=$2 inches; also, by the last example, the slant side BC$=$6·5 inches, and DC$=$5·2 inches. But,

in. in. in. in.
As 6·5 [BC] : 6 [EC] : : 5·2 [CD] : 4·8 [GC].

Likewise

in. in. in. in.
As 5·2 [CD] : 4·8 [GC] : : 4·8 [GC] 4·43077 [*ac*].

Consequently,

By subtraction CK$=$3·2 inches, and *a*K$=$1·23077 inches.

And

in. in. in. in.
As 6 [CE] : 3·2 [CK] : : 5 [AB] : $2{\cdot}66\dot{6}$ [HI].

Now the ungula *no*G above the level *ro* is equal to the cone *abc* above the same level; wherefore, if Q be put for the DIAMETER in inches, and R for the ALTITUDE in inches of a cone equal to $\frac{1}{5}$ the given cone; also S for the solidity in inches, of the cone *abc*; it will be

$$\frac{\mathrm{HI}^2 \times {\cdot}7854 \times ac}{3} - \frac{\mathrm{Q}^2 \times {\cdot}7854 \times \mathrm{R}}{3} = \mathrm{S}.$$

That is, substituting numeral values,

in. (*Const. dec.*) in. in. (*Const. dec.*) in.
$\frac{1}{3}(2{\cdot}6\dot{6}^2 \times {\cdot}7854 \times 4{\cdot}43077) - \frac{1}{3}(2{\cdot}924^2 \times {\cdot}7854 \times 3{\cdot}509)$,* or

cub. in. cub. in. cub. in.
8·2487 $-$ 7·8543$=$·3944 *the content of the cone abc, or of the ungula* GNO. But,

As Cone *a*HI : Cone *abc* : : *a*K^3 : *am*^{3}.

Hence *am*$=$·684 of an inch.

(*a*K) (*am*) (*m*K)
Lastly, 1·23077 in. $-$ ·684 in. $=$·54677 in. Ans.

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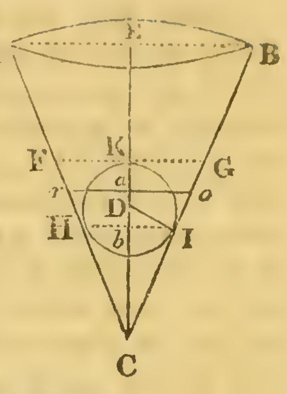
* More correctly 3·508821 inches.

(Key to Vol. II. page 265.)



Quest. 56. Let ABC (as in the two preceding Examples) represent the conical glass, *rao* the level of the water before the sphere is immersed, and FKG its surface after the ball has sunk; also let D be the center of the sphere, DI a radius at right angles to BC, and H*b*I a line parallel to FG or *ro* from the point of contact I of the sphere and glass.

Then *ac*=3·508821 inches. [*See the last Question.*]
ro =2·924 and *ao* 1·462 inches.

Also the solidity of the cone C*ro* is 7·854 cubic inches.*

Now, As C*a* : *ao* : : I*b* : *b*D. Or, in numbers,

in. in. (*b*I) (*b*D)
As 3·508821 : 1·462 : : 1 : ·41666. That is, *b*I is to *b*D in the ratio of unity to ·41666. Again,

As D*b* : *b*I : I*b* : *b*C. Consequently,

*b*C=2·4 inches, and by addition, DC=2·81666 inches.

Hence, *(still considering b*I *as unity,)* DI=1·08333 inches, CK=3·9 inches; and KG = 1·625 inches, which doubled = 3·25 inches=FG. And

$$\frac{\overset{(\mathrm{FG})}{3{\cdot}25^2} \times \overset{(\mathit{Const.\ dec.})}{{\cdot}7854} \times \overset{(\mathrm{CK})}{3{\cdot}9}}{3} = \overset{\text{inches}}{10{\cdot}78454}$$ the solidity of the cone CFG.

Likewise $(2\mathrm{DI})^3 \times {\cdot}5236$=the solidity of the immersed globe=$(2\frac{1}{6}\text{ in.})^3 \times {\cdot}5236$=5·325682 cubic inches. Now, by the Question, the solidity of the sphere immersed is equal to the solid content of the conical frustum F*ro*G. Wherefore, by subtracting this frustum from the cone CFG (above found to be 10·78454 cubic inches when DI is 1 inch) there remain 5·45886 cubic inches for the cone C*ro*; but it is known to contain 7·854 cubic inches.

By proportion, then,

cub. in. cub. in. (*sup. cone* CFG) (*tr. cone* CFG)
As 5·45886 : 7·854 : : 10·78454 : 15·5163 cubic inches, the true content of the cone CFG.

Subtract 7·854 cub. in. (the cone C*ro*).

Remainder 7·6623 solidity of the ball.

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in. (*Con. dec.*)
* *Because equal to* $\frac{1}{5}$ *the whole cone* ABC; *that is, equal to* $\frac{1}{5}(5^2 \times 7854 \times$
($\frac{1}{3}$ *Alt.*)
2 in.)=7·854 *cubic inches.*

Lastly $\sqrt[3]{\dfrac{7\cdot6623}{\cdot5236}}$ inches$=2\cdot445981$ inches. Ans.

Quest. 57. As $\overset{(Cir.)}{3\cdot1416} : \overset{(mil.\,Cir.)}{25000} :: \overset{(\text{D})}{1} : \overset{(mil.\,Di.)}{7957\cdot74}$ the diameter of the earth. And

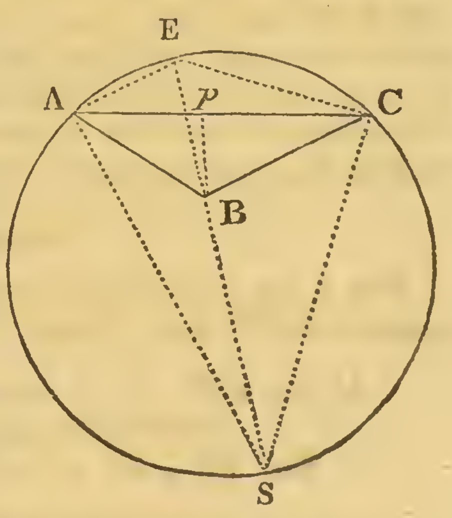
As $\overset{\text{miles}}{25000} : \overset{\text{miles}}{49\cdot5933} :: \overset{(Cir.)}{360°} : \overset{(Arc.)}{0° \,..\, 42'\cdot85}$ the angle of Oxford and London, at the center of the Earth.

Wherefore,

As Radius - - - - - - - - - - - - - - Log. ar. co. $-10\cdot$ ———
: Sec. 0° .. 42′ .. 51″ - - - - - - - - - - - - - Log. 10·0000337
:: 3978·87 miles *(the Earth's radius)* - - - Log. 3·5997597

: 3979·178 *miles, balloon's dist. from the Earth's center* - - - - - - - - - - - - *Log.* 3·5997934

But $\overset{\text{miles}}{3979\cdot178} - \overset{\text{miles}}{3978\cdot87} = \overset{\text{miles}}{\cdot308}$ or 542·08 yards. Ans.



Quest. 58.* Construct the triangle ABC having its three sides respectively equal to the sides given: namely, AB 213 yards, AC 424 yards, and BC 262 yards. Then at A make the ∠CAE equal to the observed ∠CSB 29° .. 50′; and at C make the ∠ACE equal to the observed ∠ASB 13° .. 20′.

Through the three points AEC, (E being the intersection of AE and EC) describe a circle; and join EB. If EB be produced, it will cut the opposite circumference in S, the station of the observer.

Draw AC, and CS; the positions are Geometrically determined.

Trigonometrically.

180° − 43° .. 20′ (the sum of the $\angle^{\text{les}}$ at S) $=$ 136° .. 40′ ∠AEC; whence AE, EC, Ap, pC, (p being the point in which a perp. from

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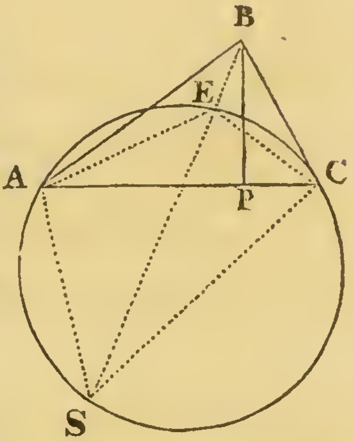
* This Question produces a figure the same in species as the figure arising from the 25th Example of HEIGHTS and DISTANCES, vol. ii. page 25. And the sides *here* are respectively to the sides *there*, as 212 to 265, which is evident by comparing AB given, in the one, with AB, in the other.

(Key to Vol. II. page 265.)

E cuts AC;) also the $\angle^{les}$ BCA, BAC, &c. &c. are all known. [*See* Ex. 25. *page* 25. *vol.* ii.]

$$\text{And, by resolution, }\left\{\begin{array}{l}\text{AS}=605{\cdot}71\text{ yards.}\\ \text{BS}=429{\cdot}68\text{ yards.}\\ \text{CS}=524{\cdot}24\text{ yards.}\end{array}\right\}\text{ Ans.}$$

(Page 266.)



Quest. 59. Construct a triangle ABC, having its three sides AB, AC, and BC, equal to the given sides, each to each: and at the point A make the ∠ CAE equal to the observed ∠ BSC 22° .. 30′: also at C make the ∠ ACE equal to the observed ∠ ASB 33° .. 45′; and from B draw a straight line to E, the point of intersection of AE and CE.

Then through the three points AEC, which are not in the same right line, describe a circle; and produce BE until it meet the opposite circumference at S, the station of the observer. If the lines SA and SC be drawn, the figure is Geometrically determined, and the distances nearly as in the Answer given.

Trigonometrically.

The ∠ AEC is the supplement of the sum of the $\angle^{les}$ at S, hence equal to 123° .. 45′. And,

As Sin. 123° .. 45′ ∠ AEC : 12 furl. AC : : Sin. 22° .. 30′ ∠ EAC : 5·523 *furlongs* EC.
: : Sin. 33° .. 45′ ∠ ACE : 8·0181 &c. *furlongs* AE.

Demitting a perpendicular from B upon AC, and proceeding in all respects as in Example 26,* page 25 of the second volume of the Course, the Answer is found as below:

$$\left.\begin{array}{l}\text{AS }10{\cdot}6545\text{ furlongs}\\ \text{BS }15{\cdot}6277\text{ furlongs}\\ \text{CS }14{\cdot}0121\text{ furlongs}\end{array}\right\}\text{ Ans.}$$

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* The present Example is in fact in all the angles the same as the Example to which we allude, and consequently the sides *here* will be to the analogous sides *there* in the ratio of 1320 yards, or 6 furlongs, BC *here*, to 400 yards BC *there* ∵ 3·3 : 1, either in yards or in furlongs.

(Key to Vol. II. page 266.)

Quest. 60. $\overset{\text{dwt.}}{5} \,..\, \overset{\text{gr.}}{9\tfrac{1}{2}} = 129\tfrac{1}{2}$ grains of standard gold in a guinea.

Now, As $\overset{s.}{21} : \overset{s.}{20} :: \overset{\text{gr.}}{129\tfrac{1}{2}} : \overset{\text{gr.}}{123\tfrac{1}{3}}$ the value of £1 in standard gold.

But, $\dfrac{\text{gr. } 123{\cdot}\dot{3} \times £480000000}{437{\cdot}5 \text{ gr. in an oz. avoird.*}} = \overset{\text{oz.}}{135314285{\cdot}7142857}$ the weight of the cube in ounces Avoirdupois.

And, As $\overset{\text{oz.}}{17724}$† : $\overset{\text{oz.}}{135314285{\cdot}7}$ &c. : : $\overset{\text{cub. ft.}}{1} : \overset{\text{cub. ft.}}{7634{\cdot}523}$ the solidity of the cube in feet.

Lastly, $\sqrt[3]{7634{\cdot}523}$ ft. $= 19{\cdot}69069$ feet. Ans.‡

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* See the First Volume of the Mathematics, page 26.

† By reference to the table of Specific Gravity, given in the Second Volume of the Course, page 231.

‡ Here we differ from Dr. Charles Hutton about 1 foot in the side of the cube, and think the gold would occupy at least *one thousand one hundred and five cubic feet more* than the space allotted it by the Doctor.

Quest. 61. $\dfrac{\overset{\text{ft.}}{20}+\overset{\text{ft.}}{22}}{2} = 21$ feet, mean breadth of the ditch.

And $\overset{\text{ft.}}{21} \times \overset{\text{ft.}}{9} \times \overset{\text{ft.}}{1000} \times 1728 = 326592000$ cubic inches, content of the ditch.

But $\overset{\text{cub. inches}}{326592000} \div \overset{\text{cub. in.}}{282} = 1158127\tfrac{31}{47}$ gallons. Ans.

Quest. 62. As $\overset{\text{miles}}{7930^3} : \overset{\text{miles}}{2160^3}$: : Earth's solidity : Moon's solid^y;

And

As $\overset{\text{miles}}{7930^2} : \overset{\text{miles}}{2160^2}$: : Earth's surface : Moon's surface.

But

$7930^3 = 49867725700.$ $\quad 2160^3 = 1007769600.$
$7930^2 = 62884900.$ $\quad 2160^2 = 4665600.$

Hence

$$\left.\begin{array}{l}\text{Solidity of the Earth : Solidity of the Moon} :: 49\tfrac{1}{2} : 1\\ \text{Surface of the Earth : Surface of the Moon} :: 13\tfrac{1}{2} : 1\end{array}\right\}\text{Ans.}$$

(Key to Vol. II. page 267.)

PRACTICAL QUESTIONS IN SPECIFIC GRAVITY.

To determine the Magnitude of a Body by its Weight.

Ex. 1. As 2520 oz. tabular specific gravity of COMMON STONE,
: 1792 oz. weight of the given block,
: : 1728 cubic inches in 1 cubic foot,
: $1228\frac{4}{5}$ *cubic inches in the block.* Ans.

Ex. 2. As 937 oz. tabular specific gravity of GUNPOWDER,
: 16 oz. weight of the quantity given,
: : 1728 cubic inches in 1 cubic foot,
: $29\frac{1}{2}$ *cubic inches* nearly. Ans.

Ex. 3. As 925 oz. tabular specific gravity of DRY OAK,
: 35840 oz. in one ton,
: : 1 cubic foot,
: $38\frac{138}{185}$ *cubic feet.* Ans.

To determine the Weight of a Body by its Magnitude.

Ex. 1. $\overset{\text{ft.}}{63}\times\overset{\text{ft.}}{12}\times\overset{\text{ft.}}{12}=9072$ cubic feet in the block.

And

As 1 cubic foot,
: 9072 cubic feet,
: : 2700 oz. tabular specific gravity of MARBLE,
: 24494400 oz. *in the block,*=683·4 *tons.* Ans.

Ex. 2. As 1728 cubic inches in 1 cubic foot,
: $35\frac{1}{4}$ cubic inches in an ale pint,
: : 937 oz. tabular specific gravity of *gunpowder,*
19·1 oz. *the weight required.*

Ex. 3. $\overset{\text{ft.}}{10}\times\overset{\text{ft.}}{3}\times\overset{\text{ft.}}{2\frac{1}{2}}=75$ solid feet of oak in the block.

(Key to Vol. II. page 267.)

And,

As 1 cubic foot,
: 75 cubic feet, content of the block,
: : 925 oz. tabular specific gravity of OAK,
: 69375 *oz.* $=4335\frac{15}{16}$ *lb. the weight required.*

*** The last six Questions are found at the 231st and following page of the second volume of the COURSE; and the three following, with only a slight variation in the first, in pages 229 and 230.

(Page 268.)

To ascertain the Specific Gravity of a Body.

lb. lb. lb.
Ex. 1. $10-6\frac{1}{6}=3\frac{5}{6}$ difference of weight in water and air.

And

As $3\frac{5}{6}$ lb. weight lost in water,
: 10 lb. weight in air,
: : 1000 oz. specific gravity of WATER,
2609 oz. *specific gravity required.*

lb. lb. lb.
Ex. 2. $18+15=33$ weight of the aggregate body in air.
$18-16=\ 2$ lost by the copper in water.
Also $33-\ 6=27$ lost by the aggregate mass in water.
And $27-\ 2=25$ difference of loss.

But, As 25 lb. : 15 lb. : : 1000 oz. : 600 oz. Ans.

(Page 269.)

To determine the Quantity of each Ingredient in a Body compounded of Two known Ingredients.

oz. oz. oz.
Ex. 1. $9000-7320=1680$ first difference.
$8784-7320=1464$ second difference.
$9000-8784=\ 216$ third difference.

Also,

$8784\times1680=14757120.$
$9000\times1464=13176000.$
$7320\times\ 216=\ 1581120.$

(Key to Vol. II. page 269.)

But,

$$\left.\begin{array}{l}\text{As } 14757120 : \overset{\text{lb.}}{112} :: 13176000 : \overset{\text{lb.}}{100} \text{ copper}\\ \phantom{\text{As }}14757120 : 112 :: 1581120 : 12 \text{ tin}\end{array}\right\} \text{Ans.}$$

Sum and Proof 112 lb.

BALLS AND SHELLS.

To determine the Weight of an Iron Ball of known Diameter.

Ex. 1. $\overset{\text{in.}}{6{\cdot}7} \times \overset{\text{in.}}{6{\cdot}7} \times \overset{\text{in.}}{6{\cdot}7} = \overset{\text{cub. inches}}{300{\cdot}763}$ cube of the diameter.

And,

lb. $\frac{9}{64} \times 300{\cdot}763 = 42{\cdot}2948$ lb. Ans.

Ex. 2. $\overset{\text{in.}}{5{\cdot}54} \times \overset{\text{in.}}{5{\cdot}54} \times \overset{\text{in.}}{5{\cdot}54} = \overset{\text{cubic inches}}{170{\cdot}031464}$ cube of the diameter.

Now,

lb. $\frac{9}{64} \times 170{\cdot}031464 = 23{\cdot}99$ lb. or 24 lb. *nearly*. Ans.

(Page 270.)

To determine the Weight of a Leaden Ball of known Diameter.

Ex. 1. $\overset{\text{in.}}{6{\cdot}6} \times \overset{\text{in.}}{6{\cdot}6} \times \overset{\text{in.}}{6{\cdot}6} = \overset{\text{cubic inches.}}{287{\cdot}496}$ cube of the diameter.

But,

lb. $\frac{3}{14} \times 287{\cdot}496 = 61{\cdot}606$ lb. Ans.

Ex. 2. $\overset{\text{in.}}{5{\cdot}3} \times \overset{\text{in.}}{5{\cdot}3} \times \overset{\text{in.}}{5{\cdot}3} = \overset{\text{cub. inches}}{148{\cdot}877}$ cube of the diameter.

And,

lb. $\frac{3}{14} \times 148{\cdot}877 = 32$ lb. *nearly*. Ans.

To ascertain the Diameter of an Iron Ball of known Weight.

Ex. 1. As $\overset{\text{lb.}}{9} : \overset{\text{lb.}}{42} :: \overset{\text{cub. in.}}{64} : \overset{\text{cub. in.}}{298{\cdot}66}$ cube of the diameter.

(Key to Vol. II. page 270.)

But $\sqrt[3]{298\cdot6\dot{6}\text{ in.}}=6\cdot685$ inches. Ans.

Or, by the Rule,

$\sqrt[3]{42\times\frac{64}{9}}=6\cdot685$ the diameter in inches, *as before.*

lb. lb. cub. in. cub. in.
Ex. 2. As 9 : 24 :: 64 : $170\cdot6\dot{6}$ cube of the diameter.

And $\sqrt[3]{170\cdot6\dot{6}\text{ in.}}=5.54$ inches. Ans.

That is,

$\sqrt[3]{24\times7\frac{1}{9}}$ lb. $=5\cdot54$ the diameter in inches, *as before.*

To ascertain the Diameter of a Leaden Ball of known Weight.

lb. lb. cub. in. cub. in.
Ex. 1. As $\frac{3}{14}$: 64 :: 1 : $298\cdot6\dot{6}$ cube of the diameter.

Now,

$\sqrt[3]{298\frac{2}{3}\text{ in.}}=6\cdot685$ inches. Ans.

lb. lb. cub. in. cub. in.
Ex. 2. As $\frac{3}{14}$: 8 :: 1 : $37\frac{1}{3}$ cube of the diameter.

Hence, by extraction of the cube root, 3·343 inches. Ans.

(Page 271.)

To determine the Weight of an Iron Shell of known diameter and thickness.

in. in. in. cub. inches
Ex. 1. $12\cdot8\times12\cdot8\times12\cdot8=2097\cdot152$ the outer diameter cubed.
And $9\cdot1\times9\cdot1\times9\cdot1=753\cdot571$ the internal diam^{r}. cubed.

1343·581 *difference* of the cubes.
Multiply by $\frac{9}{64}$

PRODUCT 188·941 lb. Ans.

(Key to Vol. II. page 271.)

$$\text{Ex. 2.}\quad \overset{\text{in.}}{9{\cdot}8} \times \overset{\text{in.}}{9{\cdot}8} \times \overset{\text{in.}}{9{\cdot}8} = \overset{\text{cub. inches}}{941{\cdot}192} \text{ the outer diameter cubed.}$$

$$\text{And } 7 \times 7 \times 7 = 343{\cdot} \quad \text{the inner diameter cubed.}$$

$598{\cdot}192$ *difference* of the cubes.

Multiply by $\frac{9}{64}$

Product $84{\cdot}12$ lb. Ans.

To determine the Quantity of Gunpowder necessary to fill a Shell of known Capacity.

$$\text{Ex. 1.}\quad \overset{\text{in.}}{9{\cdot}1} \times \overset{\text{in.}}{9{\cdot}1} \times \overset{\text{in.}}{9{\cdot}1} = \overset{\text{cub. inches}}{753{\cdot}571} \text{ cube of the internal diameter.}$$

Divide by $57{\cdot}3$ cub. inches—*the Quotient is* $13{\cdot}15$ lb. Ans.

$$\text{Ex. 2.}\quad \frac{\overset{\text{in.}}{7} \times \overset{\text{in.}}{7} \times \overset{\text{in.}}{7}}{57{\cdot}3} = 6 \text{ lb. } \textit{nearly}. \quad \text{Ans.}$$

To determine the Quantity of Powder necessary to fill a Rectangular Box of known Dimensions.

$$\text{Ex. 1.}\quad \frac{\overset{\text{in.}}{15} \times \overset{\text{in.}}{12} \times \overset{\text{in.}}{10}}{30} = 60 \text{ lb.} \quad \text{Ans.}$$

$$\text{Ex. 2.}\quad \frac{\overset{\text{in.}}{12} \times \overset{\text{in.}}{12} \times \overset{\text{in.}}{12}}{30} = 57{\cdot}6 \text{ lb.} \quad \text{Ans.}$$

To ascertain the Weight of Powder sufficient to fill a hollow Cylinder of known Length and Diameter.

$$\text{Ex. 1.}\quad \frac{(\overset{\text{in.}}{10} \times \overset{\text{in.}}{10}) \times \overset{\text{in.}}{20}}{38{\cdot}2} = 52{\cdot}35 \text{ lb.} \quad \text{Ans.}$$

(Page 272.)

$$\text{Ex. 2.}\quad \frac{(\overset{\text{in.}}{4} \times \overset{\text{in.}}{4}) \times \overset{\text{in.}}{12}}{38{\cdot}2} = 5\tfrac{5}{191} \text{ lb.} \quad \text{Ans.}$$

(Key to Vol. II. page 272.)

To determine the Size of a Shell that will just contain a given Weight of Powder.

Ex. 1. $\sqrt[3]{13\frac{1}{6}\times57{\cdot}3}=9{\cdot}1$ *nearly*, the diameter in inches. Ans.

Ex. 2. $\sqrt[3]{6\times57{\cdot}3}=7$ *nearly*, the diameter in inches. Ans.

To determine the Side of a Cubical Box that will just hold a specified Portion of Gunpowder by Weight.

Ex. 1. $\sqrt[3]{50\times30}=11{\cdot}44$, the side required, in inches.

Ex. 2. $\sqrt[3]{400\times30}=22{\cdot}89$ &c. the side required, in inches.

To determine the Length a given Weight of Gunpowder will occupy of a Cylinder of known Diameter.

Ex. 1. $\dfrac{\overset{\text{lb.}}{12}\times\overset{\text{cub. in.}}{38{\cdot}2}}{(6\frac{2}{3}\times6\frac{2}{3})\ \textit{in.}}=\dfrac{\overset{\text{cub. inches.}}{458{\cdot}4\times9}}{400\,\textit{sq. in.}}=10{\cdot}314$ in. the length reqd.

Ex. 2. $\dfrac{\overset{\text{lb.}}{20}\times\overset{\text{cub. in.}}{38{\cdot}2}}{(8\times8)\ \textit{in.}}=\dfrac{\overset{\text{cub. inches}}{764}}{64\ \textit{sq. in.}}=11\frac{15}{16}$ inches. Ans.

(Page 273.)

To ascertain the Number of Balls in a Triangular Pile.

Ex. 1. $\dfrac{30\times31\times32}{6}=4960$ balls. Ans.

Ex. 2. $\dfrac{20\times21\times22}{6}=1540$ balls. Ans.

(Page 274.)

To ascertain the Number of Balls in a Square Pile.

Ex. 1. $\dfrac{30\times31\times61}{6}=9455$ balls. Ans.

(Key to Vol. II. page 274.)

Ex. 2. $\frac{20\times 21\times 41}{6}=2870$ balls. Ans.

To ascertain the Number of Balls in a Rectangular Pile.

Ex. 1. $\overset{\text{balls}}{46}\times 3=\overset{\text{balls}}{138}$ thrice the number of shot in the length of the base of the pile. And

$15-1=14$ one ball less than the number of shot in the breadth of the base of the pile.

But $\frac{(\overset{\text{balls}}{138}-\overset{\text{balls}}{14})\times 15\times 16}{6}=4960$ balls. Ans.

Ex. 2. $\overset{\text{shot}}{59}\times 3=\overset{\text{shot}}{177}$ thrice the number of balls in the length of the base of the pile. And

$20-1=19$ one shot less than the number of balls in the breadth of the pile.

But $\frac{(\overset{\text{shot}}{177}-\overset{\text{shot}}{19})\times 20\times 21}{6}=11060$ shot. Ans.

To ascertain the Number of Balls in an Incomplete Pile.

Ex. 1. $\frac{1}{6}\times 40\times 41\times 42=11480$ shot, for the complete Pile.
And $\frac{1}{6}\times 19\times 20\times 21=1330$ shot wanting.

Difference 10150 shot. Ans.

(Page 275.)

Ex. 2. $\frac{1}{6}\times 24\times 25\times 26=2600$ shot, for the complete pile.
And $\frac{1}{6}\times 7\times 8\times 9=84$ shot wanting.

Difference 2516 shot. Ans.

Ex. 3. $\frac{1}{6}\times 24\times 25\times 49=4900$ balls, for the complete pile.
And $\frac{1}{6}\times 7\times 8\times 15=140$ balls wanting.

Difference 4760 balls. Ans.

(Key to Vol. II. page 275.)

Ex. 4. $\overset{\text{shot}}{40}-\overset{\text{shot}}{11}=\overset{\text{shot}}{29}$ in the top row of either side of the pile.

And

$20-11=\ 9$ in the top row of either end of the pile.

Therefore the length of the base of DEFECT is 28 shot, and the breadth 8 shot.

But, $(\overline{40\times3}-19)\times20\times21\times\frac{1}{6}=7070$ shot, for the whole pile.
And $(28\times3-\ 7)\times\ 8\times\ 9\times\frac{1}{6}=\ 924$ shot deficient.

Difference 6146 shot. Ans.

DISTANCE ESTIMATED BY THE VELOCITY OF SOUND.

Ex. 1. $\frac{3}{14}$mls. $\times 12=\frac{36}{14}=2\frac{4}{7}$ miles. Ans.

Or, more correctly,

As $1'' : 12'' :: \overset{\text{feet}}{1142} : \overset{\text{feet}}{13704} = \overset{\text{miles}}{2}\ ..\ \overset{\text{yds.}}{1048}$. Ans.

Ex. 2. $4''\frac{2}{3}\times8=37\frac{1}{3}$ seconds. Ans.

Ex. 3. As $4''\frac{2}{3} : 7'' :: \overset{\text{mile}}{1} : \overset{\text{mile}}{1\frac{1}{2}}$. Ans.

Ex. 4. As $\overset{\text{pulsat.}}{70} : \overset{\text{pulsat.}}{6} :: 60'' : 5''\frac{1}{7}$ time of the sound in seconds.

And

As $4''\frac{2}{3} : 5''\frac{1}{7} :: \overset{\text{mile}}{1} : \overset{\text{mile}}{1\frac{5}{49}} = \overset{\text{mile}}{1}\ ..\ \overset{\text{yds.}}{179{\cdot}6}$. Ans.*

More correctly,

The time being, as already found, $5\frac{1}{7}$ seconds, it is

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* *Differing* 18·4 *yards from Dr. Hutton's Answer, which differs* 395$\frac{1}{7}$ *yards from the truth by Rule, allowing sound to travel regularly* 1142 *feet per second. But the Student will bear in mind that, owing to currents of air and other circumstances, the velocity of sound cannot be exactly depended on.*

(Key to Vol. II. page 275.)

$$\text{As } 1'' : 5''\tfrac{1}{7} :: \overset{\text{feet}}{1142} : \overset{\text{feet}}{5873\tfrac{1}{7}} = \overset{\text{mile}}{1} : \overset{\text{yds.}}{593\tfrac{1}{7}}. \quad \text{Ans.}$$

(Page 276.)

Ex. 5. $\text{As } \overset{\text{pulsat.}}{75} : \overset{\text{pulsat.}}{5} : 60'' : 4''$ time of the sound in seconds.

But,

$\overset{\text{feet}}{1142} \times 4 = \overset{\text{feet}}{4568} = 1522\tfrac{2}{3}$ yards. Ans.

Ex. 6. $\text{As } 4''\tfrac{2}{3} : 33'' :: \overset{\text{mile}}{1} : \overset{\text{miles}}{7\tfrac{1}{14}}.$ Ans.

PRACTICAL QUESTIONS IN PHILOSOPHY.

Quest. 1. $\overset{\text{in.}}{3} \times \overset{\text{in.}}{3} \times \overset{\text{in.}}{3} \times \overset{(\textit{Const. dec.})}{\cdot 5236} = 14\cdot 1372$ cub. in. cast iron in the ball.
Multiply by ·258 lb. Avoirdupois.

PRODUCT 3·6473976 lb. Ans.

Quest. 2. $\overset{\text{in.}}{5^3} - \overset{\text{in.}}{3^3} = \overset{\text{cub. in.}}{98}$ *diff.* of the cubes of the diameters.

Now

lb. $\tfrac{9}{64} \times 98 = 13\cdot 78125$ lb. Ans.*

* *By which it appears that the weight of the Shell is more than* ½ *lb. above Dr. Hutton's estimate.*

Quest. 3. $\text{As } 4''\tfrac{2}{3} : 17'' :: \overset{\text{mile}}{1} : \overset{\text{miles}}{3\tfrac{9}{14}}.$ Ans.

Quest. 4. By Question 62, *page* 266 *of the second volume of the* COURSE, the bulk of the Earth is to that of the Moon as 49½ to 1.

(Key to Vol. II. page 276.)

$$\text{Therefore } \frac{49{\cdot}5 \times 10}{7} = 70{\cdot}7, \text{ to } 1. \quad \text{Ans.}$$

Quest. 5. Referring to the Table of Specific Gravity in vol. ii. page 231,

A cubic foot of brass weighs 8000 oz. avoirdupois.
A cubic foot of marble - - - 2700 oz.

Difference 5300 oz.
Multiply by $1\frac{1}{2}$

PRODUCT 7950 oz. = 496 lb. .. 14 oz. Ans.

Quest. 6.* 63 ft. × 12 ft. × 12 ft. = 9072 cubic feet. And

As 1 cubic foot,
: 9072 cubic feet,
: : 2700 oz. tabular specific gravity of MARBLE,
: 24494400 *oz.* = 683·4 *tons*. Ans.

Quest. 7. It is evident, since the *momenta* must be equal, that the velocities will be inversely as the weights. Hence

As 32 lb. : 10000 lb. : : 20 ft. : 6250 ft. per second. Ans.

(Page 277.)

Quest. 8. Multiplying the velocities respectively by the weights, it is

1000 (v) × 1 (w) = 1000 the momentum of the less body.
And 1 × 25 = 25 the momentum of the greater body.

But As 25 : 1000 : : 1 : 40.

Wherefore the momentum of the *less* is to that of the *greater body* as 1 to 40. Ans.

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* The present Question was surely a favourite, for this is the third time of its appearance in the second volume. See pages 232, 267, and 276.

(Key to Vol. II. page 277.)

Quest. 9. $\overset{\text{ft.}}{100}\times\overset{\text{lb.}}{20}=2000$ momentum of the 1st body.

And $\overset{(\text{M})}{2000}\div 8\text{ lb.}=250$ feet velocity of the less body per second. Ans.

Or, the velocities being in the inverse ratio of the weights, it is

As $\overset{\text{lb.}}{8}:\overset{\text{lb.}}{20}::\overset{\text{ft.}}{100}:\overset{\text{ft.}}{250}$ per second, *as before.*

Quest. 10. As $\overset{\text{lb.}}{60}:\overset{\text{lb.}}{100}::\overset{(v.)}{8}:\overset{(v.)}{13\frac{1}{3}}$ the ratio to 1, the velocity of the less body has to that of the greater. But $13\frac{1}{3}$ are to 1 as 40 to 3. Ans.

Quest. 11. $\overset{(\textit{Force})}{48}\div\overset{\text{lb.\&c.}}{8}=6$ velocity of the greater body.
And $1\div 1=1$ velocity of the less body.

Hence the velocity of the greater is to the velocity of the less, as 6 to 1. Ans.

Quest. 12. As $\overset{\text{min.}}{1}\times\overset{(v.)}{40}:\overset{\text{min.}}{120}\times\overset{(v.)}{1}::1:3$ the ratio of the space described by the swifter, to that described by the slower body. Ans.

Quest. 13. As $\overset{\text{min.}}{1}\times\overset{(v.)}{1}:\overset{\text{min.}}{12}\times\overset{(v.)}{30}::\overset{\text{ft.}}{5}:\overset{\text{ft.}}{1800}$ by the swifter body.
Subtract 5 by the slower body.

Difference 1795 feet. Ans.

Quest. 14. It is manifest that if the bodies had moved with equal velocities, the slower body would have passed over $5\times5=25$ miles.

Now, As $\overset{\text{miles}}{25}:\overset{\text{miles}}{50}::1:2$ the ratio of the time of the slower body to that of the faster. Ans.

Quest. 15. It will be,

As $2(\overset{\text{in.}}{18}\times\overset{\text{ft. high}}{104}):\overset{\text{in.}}{40}\times\overset{\text{ft. high}}{73}::\overset{\text{hrs.}}{13}:\overset{\text{hrs.}}{10}\,..\,\overset{\text{min.}}{8\frac{1}{3}}$. Ans.

(Key to Vol. II. page 278.)

Quest. 16. $\overset{\text{in.}}{100} - \overset{\text{in.}}{7\frac{1}{2}} = \overset{\text{in.}}{92\frac{1}{2}}$ dist. of the power from the fulcrum.

And

As $\overset{\text{in.}}{7\frac{1}{2}} : \overset{\text{in.}}{92\frac{1}{2}} :: \overset{\text{lb.}}{168} (=1\frac{1}{2}\text{ cwt.}) : \overset{\text{lb.}}{2072}$. Ans.

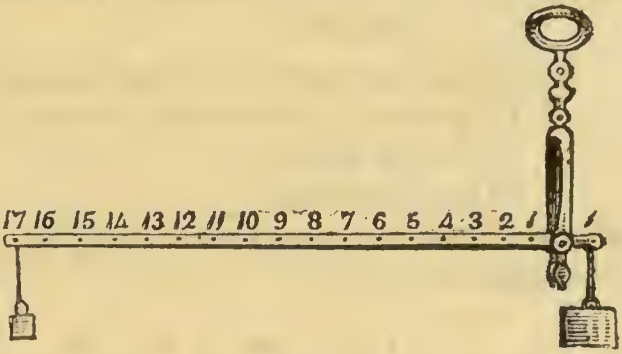
Quest. 17. $\overset{\text{lb.}}{1\frac{1}{2}} \times \overset{\text{in.}}{28} = \overset{\text{lb.}}{42}$ for the weight in the palm.
And $1\frac{1}{2} \times 12 = 18$ for the weight at the elbow.

Difference 24 lb. Ans.

Quest. 18. $\overset{\text{in.}}{70} \div \overset{\text{in.}}{2} = \overset{\text{in.}}{35}$, that is, the ratio of the distance of the power to that of the weight, from the fulcrum, is 35 to 1.

Now $9\frac{1}{2}$ cwt. $=1064$ lb.
And 1064 lb. $\div 35 = 30{\cdot}4$ lb. Ans.

Quest. 19.
As $\overset{\text{in.}}{36} : \overset{\text{in.}}{2} :: \overset{\text{lb.}}{10} : \overset{\text{lb.}}{\frac{5}{9}}$, which $\times^{\text{ed}}$ by $2 = 1\frac{1}{9}$ lb. balanced at the fulcrum: hence $8\frac{8}{9}$ lb. of the beam remain unbalanced.



But $8\frac{8}{9}\text{lb.} \times \overset{\text{in.}}{18} \times \frac{1}{2} = 80$ lb. required at the short end to keep the beam in equilibrio. And

$\overset{\text{lb.}}{95} - \overset{\text{lb.}}{80} = \overset{\text{lb.}}{15}$ the real weight to be balanced.

Now, As $\overset{\text{lb.}}{1} : \overset{\text{lb.}}{15} :: \overset{\text{in.}}{2} : \overset{\text{in.}}{30}$ the dist. of $\overset{\text{lb.}}{1}$ from the fulcrum.
2 : 15 :: 2 : 15 - - - - - - 2
3 : 15 :: 2 : 10 - - - - - - 3
4 : 15 :: 2 : $7\frac{1}{2}$ - - - - - - 4
5 : 15 :: 2 : 6 - - - - - - 5
&c. &c. &c.

Hence $\overset{\text{in.}}{30}, \overset{\text{in.}}{15}, \overset{\text{in.}}{10}, \overset{\text{in.}}{7\frac{1}{2}}, \overset{\text{in.}}{6}, \overset{\text{in.}}{4\frac{2}{7}}, \overset{\text{in.}}{3\frac{3}{4}}, \overset{\text{in.}}{3\frac{1}{3}}, \overset{\text{in.}}{3}$, &c. Ans.

Quest. 20. As $\overset{\text{ft.}}{4} : \overset{\text{ft.}}{2{\cdot}5} :: \overset{\text{lb.}}{200} : \overset{\text{lb.}}{125}$ for the shorter end. } Ans.
$4 : 1{\cdot}5 :: 200 : 75$ for the longer end. }

(Key to Vol. II. page 278.)

Quest. 21. $\sqrt{\overset{\text{lb.}}{40}\times\overset{\text{lb.}}{90}}=60$ lb. *the true weight of the body.*

But, As $\overset{\text{lb.}}{60} : \overset{\text{lb.}}{40} :: 3 : 2$ *the proportion required.*

Quest. 22. $\overset{\text{ft.}}{13}-\overset{\text{ft.}}{1}=\overset{\text{ft.}}{12}$ dist. of the power from the fulcrum, 1 foot occasioning by the Question a *diff.* of 210 lb. in the power required.

Now, As $\overset{\text{ft.}}{1} : \overset{\text{ft.}}{12} :: \overset{\text{lb.}}{210} : \overset{\text{lb.}}{2520}$. Ans.

(Page 279.)

Quest. 23. Without discussing the merits of this Example, it will be,

As 1 inch (EF) advanced towards the fulcrum by the weight,
: 12 inches (AG) advanced towards the same by the power,
:: 2 lb. the power,
: 24 lb. *the weight.*

Which is evident, since there is said to be an equilibrium in both situations.

Quest. 24. As $\overset{\text{in.}}{50} : \overset{\text{in.}}{30} :: \overset{\text{in.}}{40} : \overset{\text{in.}}{24}=$ BD.

And

As 30 in. : 40 in. :: 3 : 4 :: P : Q.

Therefore the ratio of P *to* Q *is that of* 3 *to* 4; *and the altitude* BD, 24 *inches.* Ans.

Quest. 25. $\overset{\text{ft.}}{6}+\overset{\text{ft.}}{6}=\overset{\text{ft.}}{12}=144$ inches the diameter of the circle.

And, As $\overset{\text{(Di.)}}{1} : \overset{\text{(Circ.)}}{3{\cdot}1416} :: \overset{\text{(Di.)}}{144\text{in.}} : \overset{\text{(Circumf.)}}{452{\cdot}3904\text{in.}}$ orb. of the power.
Multiply by 50 lb. the power.

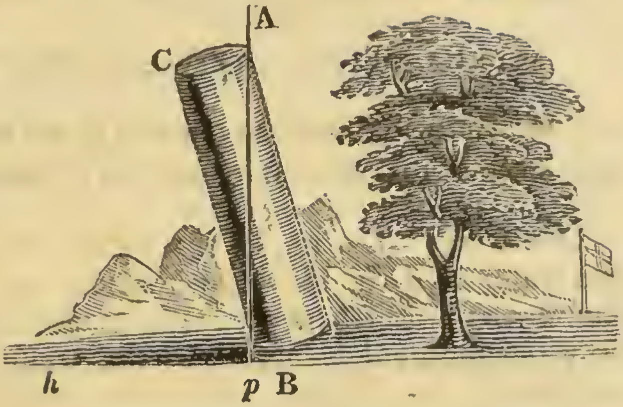
PRODUCT and Answer 22620 lb. *nearly.*

Quest. 26. As $\overset{\text{ft.}}{20} : \overset{\text{ft.}}{30} :: \overset{\text{lb.}}{150} : \overset{\text{lb.}}{225}$. Ans.

(Key to Vol. II. page 279.)

Quest. 27. As 2 in. : 12 in. : : 150 lb. : 900 lb. Ans.

Quest. 28.



Let BC represent the pillar and A*p* a perpendicular from the lowest point of the base; the ∠ A*p*C, by the question, is 15°, and AC 30 feet: wherefore it is,

As Sin. 15° ∠ A*p*C, - - - - Log. ar. co. $-10{\cdot}5870038$
: Sin. 75° ∠ *p*AC, - - - - - - - - - Log. $9{\cdot}9849438$
: : 30 feet AC, - - - - - - - - - - - - - Log. $1{\cdot}4771213$

: 111·96 *feet* *p*C, *the length required* - *Log.* $2{\cdot}0490689$

Quest. 29. $\frac{2}{5}\times 12\sqrt{\frac{4}{5}}\text{ ft.}=\frac{48}{5}\sqrt{\frac{1}{5}}\text{ ft.}=\frac{48}{25}\sqrt{5}\text{ ft.}=4{\cdot}29325056$ feet. Ans. *See page* 198, *&c. vol.* ii. *of the* MATHEMATICS.

Quest. 30. $\frac{1}{2}\times 12\sqrt{4\div 5}\text{ ft.}=12\sqrt{\frac{1}{5}}\text{ ft.}=\frac{12}{5}\sqrt{5}\text{ ft.}=5{\cdot}3665632$ feet. Ans. *See the second vol. of the* MATHEMATICS, *p.* 199.

(Page 280.)

Quest. 31.

For the rectangular wall.

Because FE* $=\text{AQ}\sqrt{\frac{m}{3n}}$ [*Prop.* xlv. *Statics*], and that

As Rad. : Sin. 60° : : 12 feet : 10·391 feet AQ, it is

$$\text{FE}=10{\cdot}391\sqrt{\frac{4}{15}}\text{ ft.}=\frac{20{\cdot}782}{15}\times\sqrt{15}\text{ ft.}=5{\cdot}365889\text{ feet.}$$

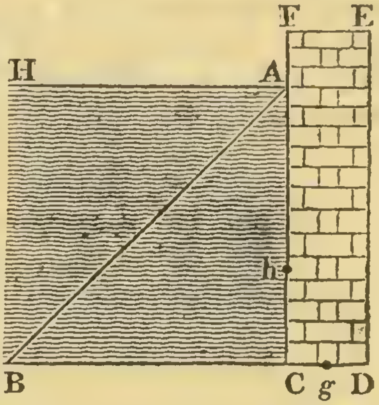
* The thickness of the wall.

(Key to Vol. II. page 280.)

And for the triangular wall.

Because FE$=$AQ$\sqrt{\frac{m}{2n}}$ [*Prop.* xlv. *Statics*], and that AQ$=10\cdot391$ feet, it is FE$=10\cdot391\sqrt{\cdot4}$ ft.$=6\cdot571845$ feet.*

Quest. 32. Let HA be the surface of the water and AC its depth, which by the Question is 10 feet. Also let FEDC be a perpendicular section of the wall, of which the height is 12 feet and thickness sought. Take CB equal to CA 10 feet, and draw the diagonal AB; the triangle ABC represents the pressure of the water against AC. [*Corol.* 3, *prop.* lxi. *Hydrostatics.*] If in CA, therefore, Ch be assumed equal to $\frac{1}{3}$ of CA, the whole pressure of the triangle ABC may be resolved to the point h. But the center of gravity of the wall is the intersection of the diagonals FD, EC, which being referred to a point g in the base, D$g=\frac{1}{2}$CD. Consequently putting x for CD the thickness sought, and considering D as the fulcrum, the weight at g, and the power at h, it is



$$\overset{(A)\dagger}{12x}\times\overset{(n)}{11}\times\overset{(Dg)}{\tfrac{1}{2}x}=\tfrac{1}{2}(\overset{(\text{AC})}{10\text{ ft.}}\times\overset{(\text{CB})}{10\text{ ft.}})\times\overset{(m)}{7}\times\overset{(Ch)}{3\tfrac{1}{3}\text{ ft.}}=1166\tfrac{2}{3}\text{ feet.}$$

That is,

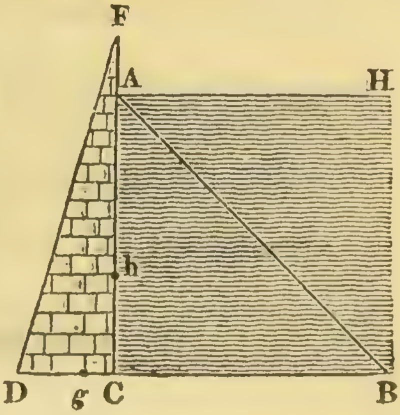
$$66x^2=1166\cdot\dot{6},\text{ or }x^2=17\cdot67\dot{6}\dot{7}.$$

Hence by extracting the square root on both sides,

$$x=4\cdot204374825\text{ \&c. feet. Ans.}$$

Quest. 33.

If FDC represent a perpendicular section of the wall, FC by the Question is 12 feet: also the water and the point h are as in the last example; but in this instance D$g=\frac{2}{3}$DC [*Art.* 236, *page* 199, *vol.* ii.] Wherefore, putting $x=$DC the thickness required, it is



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* Dr. Hutton's Answer to this part of the Question is evidently wrong, in as much as $12\sqrt{\frac{3}{10}}=6\cdot57267$ and not $6\cdot53667$.

† *Where* A *represents the area of the wall,* n *its specific gravity, and* m *the specific gravity of water.*

(Key to Vol. II. page 280.)

$$\overset{(A)}{6x} \times \overset{(n)}{11} \times \overset{(Dg)}{\tfrac{2}{3}x} = \tfrac{1}{2}(\overset{(\text{AC})}{10 \text{ ft.}} \times \overset{(\text{CB})}{10 \text{ ft.}}) \times \overset{(m)}{7} \times \overset{(Ch)}{3\tfrac{1}{3} \text{ ft.}} = 1166\tfrac{2}{3} \text{ feet.}$$

That is,

$$44x^2 = 1166{\cdot}\dot{6} \text{ feet, and } x^2 = 26{\cdot}5\dot{1}5\dot{1} \text{ feet.}$$

Hence, by extracting the square root on both sides,

$$x = 5{\cdot}1492865054 \text{ feet.} \quad \text{Ans.*}$$

☞ It is evident by the proposition quoted, that the thickness of a triangular wall must always be to that of a rectangular wall, *cæteris paribus*, as $\sqrt{\frac{3}{10}} : \sqrt{\frac{1}{5}} \because \sqrt{\cdot 3} : \sqrt{\cdot 2} \because \sqrt{3} : \sqrt{2} \because$ 17320508 : 14142136.

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* The reader is requested to examine attentively the 15th Problem (as it is termed) of the Promiscuous Exercises in the third volume of the Course, where (after noticing a Question proposed by Mr. Burton, of Salton, in 1795, and answered in the Ladies' Diary the following year by Gentlemen in different parts of the kingdom, one of whom was Dr. O. G. Gregory,) the Author gives a theorem for the thickness of a triangular wall to support a body of water of equal height; the accuracy of which theorem we dispute.

Quest. 34. As $\overset{(Times)}{\sqrt{4}} : \overset{(Time)}{\sqrt{1}} \because 2 : 1 :: \overset{\text{miles}}{95000000} : \overset{\text{miles}}{47500000}$. Ans.

Quest. 35. As $1^2 : 106^2 :: \overset{(Time)}{1} : \overset{(Times)}{11236}$.

Therefore 11236 times as hot. Ans.

Quest. 36. $95^2 \div 495^2 = \frac{9025}{245025} = \frac{1}{27}$ *nearly*. Ans.

Quest. 37. As $\overset{\text{lb.}}{10} : \overset{\text{lb.}}{112} :: \overset{S.\,di.}{1} : \overset{S.\,di.}{11{\cdot}2}$.

But, $\overset{S.\,di.}{\sqrt{11{\cdot}2}} = 3{\cdot}3466401$ semi-diameters of the earth from the earth's surface.

Or,

As $\overset{\text{lb.}}{112} : \overset{\text{lb.}}{10} \because \overset{\text{lb.}}{56} : \overset{\text{lb.}}{5} :: \overset{S.\,di.}{1} : \frac{5}{56}$ of a semidiameter of the earth from the earth's center.

Quest. 38. $\overset{\text{miles}}{7930} \div 2 = \overset{\text{miles}}{3965}$ the semidiameter of the earth.

And $\overset{\text{miles}}{3965} + \overset{\text{miles}}{50} = \overset{\text{miles}}{4015}$ distance of the body from the earth's center. It will therefore be,

As $\frac{1}{3965^2} : \frac{1}{4015^2} \because \frac{1}{15721225} : \frac{1}{16120225} :: \overset{\text{oz.}}{16} : \overset{\text{oz.}}{15} .. \overset{\text{dr.}}{9\frac{5}{8}}$ *nearly*. Ans.

(Key to Vol. II. page 280.)

Quest. 39. As $\overset{\text{miles}}{2160^3} : \overset{\text{miles}}{7930^3} :: 1 : 49{\cdot}5$ the ratio of bulk.

Now $49{\cdot}5 \times 10 = 495$ quantity of matter in the earth.

And $1 \times 7 = 7$ - - - - - - - - - - in the moon.

But, As $495+7 : 495 :: \overset{\text{Di.}}{30} : \overset{\text{Di.}}{29\tfrac{292}{505}}$.

And $\overset{\text{Di.}}{30} - \overset{\text{Di.}}{29\tfrac{292}{502}} = \overset{\text{Di.}}{\tfrac{210}{502}} = \tfrac{105}{251}$ diameters of the earth from the earth's center.

Also $\overset{\text{Di.}}{\dfrac{251^*}{502}} - \overset{\text{Di.}}{\dfrac{210^\dagger}{502}} = \overset{\text{Di.}}{\dfrac{41}{502}} = 647{\cdot}669$ miles below the surface of the earth. Ans.

* That is, one semidiameter of the earth.

† A fraction equal to $\tfrac{105}{251}$ diameters, as above.

(Page 281.)

Quest. 40. Put x for the distance of the point sought, in diameters of the earth from the earth; a for the quantity of matter in our own globe; and b for that in the moon; also let d be the mutual distance of the orbs in diameters of the earth;

Then, As $10a : 7b :: x^2 : \overline{d-x}|^2$.

Whence $x = 26\tfrac{9}{11}$. Therefore the point required is $26\tfrac{9}{11}$ of the earth's diameters from the earth's center, and $3\tfrac{2}{11}$ times the diameter of the earth from the center of the moon.

Quest. 41. As $\overline{1''}|^2 : \overline{11''}|^2$, that is, As $1 : 121 :: 16\tfrac{1}{12}$ feet : $1946\tfrac{1}{12}$ feet. Ans.

Quest. 42. As $\overline{1''}|^2 : \overline{6''}|^2$, that is, As $1 : 36 :: 16\tfrac{1}{12}$ ft. : 579 ft. the depth of the 6″ well.

And As $\overline{1''}|^2 : \overline{10''}|^2$, that is, As $1 : 100 :: 16\tfrac{1}{12}$ ft. : $1608\tfrac{1}{3}$ feet, the depth of the 10″ well.

But $1608\tfrac{1}{3}$ ft. $-$ 579 ft. $= 1029\tfrac{1}{3}$ feet *difference*. Ans.

Quest. 43. As $\overline{1''}|^2 : \overline{19\tfrac{1}{2}''}|^2$, that is, As $1 : 380{\cdot}25 :: 16\tfrac{1}{12}$ feet : $6115\tfrac{11}{16}$ feet. Ans.

(Key to Vol. II. page 281.)

Quest. 44. As $16\frac{1}{12}$ ft. : 400 ft. :: $1''$: $25''$, hence $5''$ nearly. Ans.

Quest. 45. $1''$ $2''$ $3''$ &c. *series in seconds.*
$16\frac{1}{12}$ ft. $64\frac{1}{3}$ ft. $144\frac{3}{4}$ ft. spaces passed through at the end of each second.

Also, $1''$ $2''$ $3''$ &c. *series in seconds.*
$16\frac{1}{12}$ ft. $48\frac{1}{4}$ ft. $80\frac{5}{12}$ ft. spaces for each second.

Likewise, 100 ft. $-$ $80\frac{5}{12}$ ft. $=$ $19\frac{7}{12}$ ft.

But, As $80\frac{5}{12}$ ft. : $19\frac{7}{12}$ ft. :: $3''$: $\cdot6088''$.

$$\left.\begin{array}{l}\text{Again, } 3''+\cdot6088=3\cdot6088'' \text{ the whole time of falling.}\\ \text{Lastly, As } \overline{1''}|^2 : \overline{3\cdot6088''}|^2 :: 16\cdot08\dot{3}\text{ft.} : 209\cdot4559443475 2\text{ft.}\end{array}\right\}\text{Ans.}$$

Quest. 46. If x be put for the number of seconds the stone is in motion, it will be by the question

$$16\tfrac{1}{12}x^2+1142x=1142\text{ ft.}\times10''=11420. \quad \text{That is,}$$

$$x^2+\frac{1142x}{16\cdot083}=\frac{11420}{16\cdot083}; \quad \text{or, reduced, } x^2+71x=710. \text{ nearly.}$$

Hence $x=8''\cdot8876$.

But, As $\overline{1''}|^2$: $\overline{8''\cdot8876}|^2$ :: $16\frac{1}{12}$ ft. : $1270\cdot3865$ ft. Ans.

(*Cir.*) (*Di*) in. in.
Quest. 47. As $3\cdot14159$: 1 :: 193 : $19\cdot6$.

And $19\cdot6$ in. $\times$ $2=39\cdot2$ in. or more correctly $39\frac{1}{8}$ in. Ans.

(Page 282.)

Quest. 48.

$$\left.\begin{array}{l}\text{As } \overline{1''}|^2 : \overline{2''}|^2 :: 39\tfrac{1}{8}\text{in.} : 156\tfrac{1}{2} \text{ in. length of a pendl. of } 2''\\ \quad \overline{1''}|^2 : \overline{\tfrac{1}{2}''}|^2 :: 39\tfrac{1}{8}\text{in.} : \quad 9\tfrac{25}{32} \text{ inches - - - - - - - - } \tfrac{1}{2}''\\ \quad \overline{1''}|^2 : \overline{\tfrac{1}{4}''}| \;\; :: 39\tfrac{1}{8}\text{in.} : \quad 2\tfrac{57}{128} \text{ inches - - - - - - - - } \tfrac{1}{4}''\end{array}\right\}\text{Ans.}$$

in. in. (*Sq. vib. per m.*)
Quest. 49. As 12 : $39\frac{1}{8}$ :: 3600 : $11737\cdot5$ sqr. of vibr. per min.

And, As 6 : $39\frac{1}{8}$ :: 3600 : $23475\cdot$ sqr. of vibr. per min.

Now $\sqrt{11737\cdot5}=108\cdot33$ vibr. per min. by a pendl. of 1 foot.
And $\sqrt{23475\cdot0}=153\cdot21$ vibr. per min. by a pendl. of $\frac{1}{2}$ foot.

(Key to Vol. II. page 282.)

Again, $\overset{\text{vibr.}}{153{\cdot}21} \times 60' = 9192{\cdot}6$ vibrations in an hour.

And, $108{\cdot}33 \times 60' = 6499{\cdot}8$ vibrations in an hour.

Difference $\underline{\overline{2692{\cdot}8}}$. Ans.

Quest. 50. As $\overset{\text{in.}}{18} : \overset{\text{in.}}{39\frac{1}{8}} :: 3600 : \overset{(\textit{Sq. vib. per m.})}{7825}$ square of vibr. per min.

But, $\sqrt{7825} = 88{\cdot}4$ vibrations per minute.

And, $88{\cdot}4 \div 60 = 1{\cdot}473$ vibrations per second.

Also, 8 vib. $\div 1{\cdot}473 = 5''{\cdot}4$ time marked by the pendulum.

Now putting x for the time of the fall in seconds, there arises the following equation:

$16\frac{1}{12}x^2 + 1142x = 1142\,\text{ft.} \times 5''{\cdot}4 = 6167$. That is,

$x^2 + 71x = 383{\cdot}44$. Whence $x = 5''{\cdot}04$.

But $16\frac{1}{12}\,\text{ft.} \times 5''{\cdot}04 \times 5''{\cdot}04 = 408{\cdot}54$ feet; or (by keeping several more decimal places from the beginning of the operation) 412·61 feet. Ans.

Quest. 51. As Rad. - - - Log. ar. co. $-10{\cdot}0$

: Co. Sin. 10° giv. ∠, - Log. 9·9933515

:: 20 feet, - - - - - - - Log. 1·3010300

: 19·696 *feet* - - - - - *Log.* 1·2943815

Now 20 ft. − 19·696 ft. = ·304 ft. versed Sine of 10° (to 20 ft. Rad.)

But, As $\sqrt{16\frac{1}{12}\,\text{ft.}} : \sqrt{{\cdot}304\,\text{ft.}} :: 32\frac{1}{6}\,\text{ft.}$ 4·4213 ft. Ans.

Quest. 52. As $\overset{\text{ft.}}{16\frac{1}{12}} : \overset{\text{ft.}}{10} :: \overline{1''}^{\,2} : {\cdot}62176$ square of the time in seconds for the perpendicular altitude.

Now, $\sqrt{{\cdot}62176''} = {\cdot}78852''$ the time for 10 feet.

$$\left.\begin{array}{l}\text{And, As } 10\,\text{ft.} : 100\,\text{ft.} :: {\cdot}78852'' : 7''{\cdot}8852 \text{ the time reqd.}\\ \text{Also } \dfrac{100 \times 2}{7''{\cdot}8852} = 25{\cdot}364 \text{ feet per second velocity acqd.}\end{array}\right\} \text{Ans.}$$

Quest. 53. $\dfrac{\overset{\text{vib.}}{60^2} \times \overset{\text{in.}}{39\frac{1}{8}}}{(40 \times 40)\,\textit{vib.}} = 88\frac{1}{32}$ inches, the distance of the center of oscillation from the axis of motion.

(Key to Vol. II. page 282.)

And, As $\overset{\text{in.}}{118} : \overset{\text{in.}}{88}$* $:: \overset{\text{in.}}{30} : \overset{\text{in.}}{22{\cdot}36}$ the length of the chord to radius 88 inches. Referring now to Prop. lviii. of Statics, [*See pp.* 219, 220, and 221, *vol.* ii.]

$$\left.\begin{array}{lcl} a & = & 16\tfrac{1}{12} \text{ feet.} \\ b & = & 1 \text{ lb.} \\ c & = & 22{\cdot}36 \text{ in.} \end{array}\right| \quad \left.\begin{array}{c} g \\ i \\ o \end{array}\right\} = 88 \text{ inches.}$$

$$\text{and } p = 500 \text{ lb.}$$

Therefore,

$$\frac{bii+gop}{bio} \times c \times \sqrt{\frac{2a}{o}} = 501 \times \frac{22{\cdot}36 \text{ in.}}{12 \text{ in.}} \times 2{\cdot}094 = \overset{\text{ft.}}{933{\cdot}53} \times 2{\cdot}094 =$$

1954·81182 *feet per second, the velocity of the* BALL. And $\frac{ci}{o}\sqrt{\frac{2a}{o}} = 1{\cdot}86\dot{3}\text{ ft.} \times 2{\cdot}094 = 3{\cdot}9018199$ *feet, the velocity with which the center of oscillation passes the perpendicular.*

Quest. 54. As $\overset{\text{oz.}}{1000} : \overset{\text{oz.}}{925} :: \overset{\text{in.}}{12} : 11\tfrac{1}{10}$ inches. Ans.

Quest. 55. Because the area of a sphere of which the diameter is 1 foot$=18{\cdot}8496$ inches, [Prob. vii. page 49. vol. ii.] it is $18{\cdot}8496 \times \tfrac{12}{2} = 113{\cdot}097\dot{6}$ inches.

And $\sqrt{113{\cdot}0976} = 10{\cdot}6347$ inches. Now,

As 1000 oz. : 925 oz. : : 10·6347 inches : 9·9867 inches. Ans.

(Page 283.)

Quest. 56. Let x^3 be solid content of the wood in inches.

Then, As $x^3 : (x^3 - 3x^2) :: 1000$ oz. : $(1000 - \frac{3000}{x})$ oz. the specific gravity of the wood.

Also, As $x^3 : (x^3 - 4\tfrac{8}{103}x^2) :: 1030$ oz. : $(1030 - \frac{4200}{x})$ oz. the specific gravity of the same wood. Therefore $30x = 1200$,

Or, $x = 40$. But $(1000 - \tfrac{3000}{40})$ oz.$=925$ oz.

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* The fraction is omitted as of comparatively small value, but this omission occasions a little deviation in the Answer. The data were not sufficiently clear to determine correctly the center of gravity which might have been higher than the center of oscillation, and hence the center of gyration not precisely 88 inches.

(Key to Vol. II. page 283.)

Now in the table of specific gravity 925 oz. correspond to OAK. And it is evident that each side of the cube is 40 inches. Q. E. I.

$$\text{Quest. 57.}\quad \overset{\text{oz.}}{12} \div (\overset{\text{oz.}}{12} - \overset{\text{oz.}}{7}) = 2\tfrac{2}{5}$$

$$\text{And}\quad 14\tfrac{1}{2} \div (14\tfrac{1}{2} - 9) = 2\tfrac{7}{11}$$

$$\text{But, As } 2\tfrac{2}{5} : 2\tfrac{7}{11} :: 145 : 132. \quad \text{Ans.}$$

$$\text{Quest. 58.}\quad \overset{\text{gr.}}{171} \div (\overset{\text{gr.}}{171} - \overset{\text{gr.}}{120}) = \overset{\text{gr.}}{3\tfrac{2}{9}} \text{ glass.}$$

$$\text{And}\quad 102 \div (102 - 79) = 4\tfrac{10}{23} \text{ magnet.}$$

$$\text{But, As } \overset{\text{gr.}}{3\tfrac{2}{9}} : \overset{\text{gr.}}{4\tfrac{10}{23}} :: \overset{(\textit{Glass})}{10} : \overset{(\textit{Magnet})}{13} \textit{ nearly.} \quad \text{Ans.}$$

$$\text{Quest. 59.}\quad \overset{\text{oz.}}{63} \div \overset{\text{oz.}}{10\cdot36} = \overset{\text{cub. inches}}{6\cdot08108} \text{ had the crown been gold.}$$

$$63 \div 5\cdot85 = 10\cdot76923 \text{ - - - - - - - - - - silver.}$$

$$\text{By alligation, } \overset{\text{cub. in.}}{8\cdot2245} \left\{ \begin{matrix} \overset{\text{cub. inches}}{6\cdot08108} \\ 10\cdot76923 \end{matrix} \right\} \begin{matrix} \overset{\text{cub. inches}}{2\cdot54473} \text{ cubic inches.} \\ 2\cdot14342 \text{ cubic inches.} \end{matrix}$$

4·68815 Sum

$$\text{By division, } \overset{\text{cub. in.}}{4\cdot68815} \left\{ \begin{matrix} \overset{\text{cub. inches}}{2\cdot54473} \\ 2\cdot14342 \end{matrix} \right\} \begin{matrix} \overset{\text{cub. in.}}{\cdot5428} \text{ gold.} \\ \cdot4572 \text{ silver.} \end{matrix}$$

$$\text{But } \cdot5428 \times 63\text{oz.} = 34\cdot1884 \text{ oz. gold} = \text{lb. } 2 \,..\, \overset{\text{oz.}}{10} \,..\, \overset{\text{dwt.}}{3} \,..\, \overset{\text{gr.}}{22\tfrac{1}{4}} \text{ gold.}$$

$$\text{And } \cdot4572 \times 63\text{oz.} = 28\cdot8036 \text{ oz. silver} = \text{lb. } 2 \,..\, 4 \,..\, 16 \,..\, 1\tfrac{3}{4} \text{ silver.}$$

NOTE.—*Although this solution is generally considered correct, yet in point of fact it is erroneous; for an alloy of gold and silver has less bulk than the sum of the bulks of the two metals before the alloy: and consequently Archimedes was deceived when he thought he had discovered the true method of examining the crown. Independent, indeed, of the above circumstance, much depends on the temperature of the atmosphere and water at the time of observation.*

Quest. 60. 3·84 lb. × 12 ÷ 1·4921 oz. = 30·816 cubic inches of glass in the bottle.

And 231 cubic inches = 1 gallon, the given quant. of brandy.

Also 124·0008 oz. = weight of the brandy.

The bottle and brandy therefore weigh 170·0808 oz.

- - - - - - - - - - - - - - - - - contain 261·816 cub. in.

(Key to Vol. II. page 283.)

But $261{\cdot}816 \times {\cdot}59542$ oz. $= 155{\cdot}89048272$ oz. the weight of an equal bulk of salt water.

Lastly, $170{\cdot}0808$ oz. $- 155{\cdot}89048272$ oz. $= 14{\cdot}1903728$ oz. Ans.

Quest. 61. $\overset{\text{cub. in.}}{231} \times 5 = 1155$ cubic inches of brandy.

And $1728 \div 8 = 216$ cubic inches of oak in the cask.

Also 925 oz. $\div 8 = 115{\cdot}625$ oz. weight of the wood.

But 1371 cubic inches = the content of the $\frac{1}{2}$ anker and brandy.

And $1371 \times {\cdot}59542$ oz. $= 816{\cdot}32082$ oz. the gravity of an equal bulk of salt water.

Now the brandy weighs 620·04 oz.; hence 80·691 *absolute* oz. are necessary to sink the cask.

Again,

As the wt. of lead : the wt. of salt water : : 11325 oz. : 1030 oz.

Consequently $\frac{1}{10{\cdot}99514}$ is the proportion of loss of weight lead sustains in salt water.

But,

$$\overset{\text{oz.}}{80{\cdot}691} - \frac{1}{10{\cdot}99514} \text{ of } \overset{\text{oz.}}{80{\cdot}691} : \overset{\text{oz.}}{80{\cdot}691} :: \overset{\text{oz.}}{80{\cdot}691} : \overset{\text{oz.}}{89{\cdot}743}. \quad \text{Ans.}$$

Quest. 62. As $\overset{\text{cub. ft.}}{1} : \overset{\text{cub. ft.}}{50000} :: 1000 \text{ oz.} : 1395\frac{5}{56}$ tons. Ans.

(Page 284.)

Quest. 63. $\overset{\text{in.}}{30} \times 14 \div 12 = 35$ feet, height of water.

And 1000 oz. $\div 1{\cdot}22\dot{2}$ (specif. gr. of air) $= 818{\cdot}\dot{1}\dot{8}$ the ratio to 1, that the weight of water bears to that of air; hence

35 ft. $\times 818{\cdot}\dot{1}\dot{8} = 28818{\cdot}1818$ feet $= 5{\cdot}458$ miles *nearly*, the height of air. Ans.[*]

(*See art.* 368, *page* 243, *vol.* ii.)

Quest. 64. As $\overset{\text{ft.}}{16\frac{1}{12}} : a :: \overline{\overset{\text{ft.}}{32\frac{1}{6}}}\,^2 : v.^2$ of mercury per second

As $16\frac{1}{12} : b :: \overline{32\frac{1}{6}}\,^2 : v.^2$ of water per second

As $16\frac{1}{12} : c :: \overline{32\frac{1}{6}} \;\; : v.^2$ of air per second

where $a = 30$ inches, $b = 35$ feet, and $c = 5{\cdot}5240$ miles.

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[*] *Differing from Dr. Hutton more than* 348 *feet in the height of the atmosphere, though agreeing with him in the water-barometer.*

(Key to Vol. II. page 284.)

Therefore

The velocity of quicksilver is 12·681 ft. per second.
The velocity of water - - - 47·447 ft. per second.
The velocity of air - - - 1369·8 ft. per second.

(*See pages* 233, 234, *and* 235, *vol.* ii.)

ft. ft. ft.
Quest. 65. $10-1=9$ and $1\times9\times4=36$ but $\sqrt{36}=6·$
$10-2=8$ - - $2\times8\times4=64$ - - $\sqrt{64}=8·$
$10-3=7$ - - $3\times7\times4=84$ - - $\sqrt{84}=9·16515$
$10-4=6$ - - $4\times6\times4=96$ - - $\sqrt{96}=9·79796$
$10-5=5$ - - $5\times5\times4=100$ - - $\sqrt{100}=10·$
$10-6=4$ - - $6\times4\times4=96$ - - $\sqrt{96}=9·79796$
$10-7=3$ - - $7\times3\times4=84$ - - $\sqrt{84}=9·16515$
$10-8=2$ - - $8\times2\times4=64$ - - $\sqrt{64}=8·$
$10-9=1$ - - $9\times1\times4=36$ - - $\sqrt{36}=6·$

Put b for the height of the fluid above the orifice, in inches; $a=·031416$ of a square inch the area of the orifice; and let $g=193$ inches the descent in the first second, of a heavy body falling freely at the surface of the earth. Then

$a\times2\sqrt{gb}=2a\sqrt{gb}$; which taken for the several heights, the result is,

cubic inches
3·0238 per second discharged through the hole 9 feet high.
4·2763 - - - - - - - - - - - - - 8 feet high.
5·2374 - - - - - - - - - - - - - 7 feet high.
6·1884 - - - - - - - - - - - - - 6 feet high.
6·7613 - - - - - - - - - - - - - 5 feet high.
7·4067 - - - - - - - - - - - - - 4 feet high.
8·0002 - - - - - - - - - - - - - 3 feet high.
8·5525 - - - - - - - - - - - - - 2 feet high.
9·0713 - - - - - - - - - - - - - 1 foot high.

Sum $58·5179\times60''\times10'\div282$ cub. in. $=124·5$ gallons. Ans.*

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* This Answer differs about $\frac{3}{5}$ of a gallon from the Answer given with the Question.

(Page 285.)

(*Di.*) (*Circumf.*) (*Di.*) (*Cir.*)
Quest. 66. As 1 : 3·1416 : : 100 ft. : 314·16 feet the circumference of the globe.

(Key to Vol. II. page 285.)

And 314·16 ft. × 100 ft. = 31416 sq. feet the surface of the globe.

This $\times^{\text{ed}}$ by 144 = 4523904 square inches of surface.

Again, $\overset{\text{ft.}}{100} \times \overset{\text{ft.}}{100} \times \overset{\text{ft.}}{100} \times \overset{(\textit{Const. de.})}{\cdot 5236} = 523600$ cubic feet contained.

Now, As $\overset{\text{cub. ft.}}{1} : \overset{\text{cub. ft.}}{523600} :: \overset{\text{oz.}}{1\frac{2}{9}} : \overset{\text{oz.}}{639956}$ of air displaced.

And, As 9000 oz. : 639956 oz. :: 1 cub. ft. : $71\cdot106\dot{2}$ cubic feet of copper in the globe.

But 71·1062 cubic feet = 71·1062 square feet 12 inches thick.

Therefore (reciprocally) $\overset{\text{sq. ft.}}{31416} : \overset{\text{inches thick}}{12} :: \overset{\text{sq. ft.}}{71\cdot1062} : \cdot0271$ of an inch. Ans.*

Quest. 67. As $\overset{(\textit{Di.})}{1} : \overset{(\textit{Circ.})}{3\cdot1416} :: \overset{(\textit{Di.})}{100\text{ ft.}} : \overset{(\textit{Circ.})}{314\cdot16\text{ ft.}}$ circumf. of the balloon.

Also 314·16 ft. × 100 ft. = 31416 square feet of surface.

But $\frac{31416}{100\times12} = 26\cdot18$ cubic feet of copper in the balloon.

Now, As $\overset{\text{c. ft.}}{1} : \overset{\text{c. ft.}}{26\cdot18} :: \overset{\text{oz.}}{9000} : \overset{\text{oz.}}{235620}$ the weight of copper in the balloon.

And $\overset{\text{ft.}}{100} \times \overset{\text{ft.}}{100} \times \overset{\text{ft.}}{100} \times \overset{(\textit{Const. de})}{\cdot 5236} = 523600$ cubic feet in the sphere.

Also, As $\overset{\text{c. ft.}}{1} : (\tfrac{1}{10} \text{ of } 1\tfrac{2}{9})$ oz. :: $\overset{\text{c. ft.}}{523600} : 63995\cdot6$ oz. of inflam. air.

And, As $\overset{\text{c. ft.}}{1} : 1\frac{2}{9}$ oz. :: $\overset{\text{cub. ft.}}{523600} : 639956$ oz. of atmospheric air displaced.

But 235620 oz. (wt. of the copp.) + 63995·6 oz. (wt. of the gas) = 299615·6 oz. weight of the inflated balloon.

Therefore

639956 oz. − 299615·6 oz. = 340340·4 oz. = 21271·28 lb. Ans.*

Quest. 68. $29\frac{1}{2}$ inches × 14 = $34\cdot41\dot{6}$ feet = 413 inches, the height to which water ought to rise in a Toricellian tube.

And 413 in. + 30 in. = 443 inches.

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* Which is nearly, but not exactly, Dr. Hutton's Answer.

* Being about $1\frac{3}{4}$ lb. less than given in the COURSE.

(Key to Vol. II. page 285.)

Now if x represent the height sought in inches, it will be
As $443-x : 413 :: 36 : 36-x$.

Therefore $x^2-479x=-1080$ inches.

Hence $x=239{\cdot}5\mp237{\cdot}2345885=2{\cdot}2654115$. Ans.

Quest. 69. $\overset{\text{ft.}}{8}\times\overset{\text{ft.}}{8}\times\overset{(\textit{Con. de.})}{{\cdot}7854}\times\overset{(\frac{1}{2}\textit{Alt.})}{4}=\overset{\text{cub. ft.}}{201{\cdot}0624}$ cont. of the Paraboloid.

Also $30{\cdot}9$ inches $\times\,14=432{\cdot}6$ inches $=36$ feet *nearly* for the altitude of water in vacuo.

Likewise 5 fathoms $=30$ ft.; 10 fath^{s}. $=60$ ft.; 15 fath^{s}. $=90$ ft. and 20 fath^{s}. $=120$ ft.

But, As $\overset{\text{ft.}}{36}+\overset{\text{ft.}}{30}:\overset{\text{ft.}}{36}::\overset{\text{cub. feet}}{201{\cdot}0624}:\overset{\text{cub. feet}}{109{\cdot}6704}$ at 5 fathoms.
$36+60:36::201{\cdot}0624:75{\cdot}3984$ at 10 fathoms.
$36+90:36::201{\cdot}0624:54{\cdot}8352$ at 15 fathoms.
$36+120:36::201{\cdot}0624:46{\cdot}4$. at 20 fathoms.

Again, $\overset{\text{cub. ft.}}{109{\cdot}6704}\div\overset{(\textit{Const. dec.})}{{\cdot}7854}=139{\cdot}63\dot{6}\dot{3}$ - - 5 fathoms.
$75{\cdot}3984\div{\cdot}7854=96{\cdot}0000$ - - 10 fathoms.
$54{\cdot}8352\div{\cdot}7854=69{\cdot}8181$ - - 15 fathoms.
$46{\cdot}4000\div{\cdot}7854=59{\cdot}0769$ - - 20 fathoms.

And $\frac{1}{2}\sqrt{139{\cdot}63\dot{6}\dot{3}}=5{\cdot}90845$ ft. depth of air within at 5 fath.
$\frac{1}{2}\sqrt{96{\cdot}0000}=4{\cdot}89898$ ft. - - - - - - - - - - 10 fath.
$\frac{1}{2}\sqrt{69{\cdot}8181}=4{\cdot}17786$ ft. - - - - - - - - - - 15 fath.
$\frac{1}{2}\sqrt{59{\cdot}0769}=4{\cdot}15692$ ft. - - - - - - - - - - 20 fath.

But $\overset{\text{ft.}}{8}-\overset{\text{feet}}{5{\cdot}90845}=\overset{\text{feet}}{2{\cdot}09154}$ height of water within, at 5 fathoms.
$8-4{\cdot}89898=3{\cdot}10101$ - - - - - - - - - - - - 10 fathoms.
$8-4{\cdot}17786=3{\cdot}82213$ - - - - - - - - - - - - 15 fathoms.
$8-4{\cdot}15692=4{\cdot}15692$ - - - - - - - - - - - - 20 fathoms.
Ans.*

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* *Differing the tenth part of an inch at the third depth, and considerably less at the other depths, from the altitude found by* Dr. Hutton.

END OF THE PRACTICAL EXERCISES IN PHILOSOPHY.

FLUXIONS.

(Key to Vol. II. page 299.)

PRACTICAL EXAMPLES IN FINDING FLUXIONS.

Ex. 1. The fluxion of a being 0 (*page* 289), it is $a\dot{x}y+ax\dot{y}$, or $a(\dot{x}y+x\dot{y})$. Ans.

Ex. 2. The fluxion of xyz being $\dot{x}yz+x\dot{y}z+xy\dot{z}$, it is $b\dot{x}yz+bx\dot{y}z+bxy\dot{z}$, or $b(\dot{x}yz+x\dot{y}z+xy\dot{z})$. Ans.

Ex. 3. $cx\times(ax-cy)=acx^2-c^2xy$; but the fluxion of acx^2 is $2acx\dot{x}$, and the fluxion of c^2xy is $c^2\dot{x}y+c^2x\dot{y}$.

Therefore $2acx\dot{x}-(c^2\dot{x}y+c^2x\dot{y})=c(2ax\dot{x}-c\dot{x}y-cx\dot{y})$. Ans.

Ex. 4. Ans. $mx^{m-1}y^n\dot{x}+ny^{n-1}x^m\dot{y}=(my\dot{x}+nx\dot{y})x^{m-1}y^{n-1}$.

Ex. 5. Ans. $mx^{m-1}y^nz^r\dot{x}+ny^{n-1}x^mz^r\dot{y}+rz^{r-1}x^my^n\dot{z}$.

Ex. 6. $(x+y)\times(x-y)=x^2-y^2$.

Therefore $2x\dot{x}-2y\dot{y}$. Ans.

Ex. 7. Here the fluxion of $x^2=2x\dot{x}$

Multiply by $2a$

PRODUCT $4ax\dot{x}$. Ans.

Ex. 8. The fluxion of x^3 is $3x^2\dot{x}$

Multiply by 2

PRODUCT $6x^2\dot{x}$. Ans.

(Key to Vol. II. page 299.)

Ex. 9. The fluxion of x^4y is $4x^3y\dot{x} + x^4\dot{y}$
Multiply by 3

PRODUCT $12x^3y\dot{x} + 3x^4\dot{y}$. Ans.

Ex. 10. The fluxion of $x^{\frac{2}{3}}y^4$ is $\frac{2}{3}x^{-\frac{1}{3}}y^4\dot{x} + 4x^{\frac{2}{3}}y^3\dot{y}$
Multiply by 4

PRODUCT $2\frac{2}{3}x^{-\frac{1}{3}}y^4\dot{x} + 16x^{\frac{2}{3}}y^3\dot{y}$. Ans.

Ex. 11. The fluxion of ax^2y is $2axy\dot{x} + ax^2\dot{y}$. And the fluxion of $x^{\frac{1}{2}}y^3$ is $\frac{1}{2}x^{-\frac{1}{2}}y^3\dot{x} + 3x^{\frac{1}{2}}y^2\dot{y}$. Therefore

$$2axy\dot{x} + ax^2\dot{y} - \tfrac{1}{2}x^{-\frac{1}{2}}y^3\dot{x} - 3x^{\frac{1}{2}}y^2\dot{y}, \text{ or}$$

$$\left(2axy - \frac{\frac{1}{2}y^3}{\sqrt{x}}\right)\dot{x} + (ax^2 - 3\surd xy^2)\dot{y}. \quad \text{Ans.}$$

Ex. 12. The fluxion of $4x^4$ is $16x^3\dot{x}$.
- - - - - - - of x^2y is $2xy\dot{x} + x^2\dot{y}$. And
- - - - - - - of $3byz$ is $3b\dot{y}z + 3by\dot{z}$. Therefore
$(16x^3 - 2xy)\dot{x} + (3bz - x^2)\dot{y} + 3by\dot{z}$. Ans.

Ex. 13. Ans. $\frac{1}{n}x^{\frac{1-n}{n}}\dot{x}$.

Ex. 14. Ans. $\frac{m}{n}x^{\frac{m-n}{n}}\dot{x}$.

Ex. 15. Ans. $-\frac{m}{n}x^{\frac{-m+n}{n}}\dot{x} = \frac{-\frac{m}{n}\dot{x}}{x^{\frac{m+n}{n}}}$.

Ex. 16. Ans. $\frac{1}{2}x^{-\frac{1}{2}}\dot{x} = \frac{\frac{1}{2}\dot{x}}{\sqrt{x}} = \frac{\dot{x}}{2\sqrt{x}}$.

Ex. 17. Ans. $\frac{1}{3}x^{-\frac{2}{3}}\dot{x} = \frac{\frac{1}{3}\dot{x}}{\sqrt[3]{x^2}} = \frac{\dot{x}}{3x^{\frac{2}{3}}}$.

(Key to Vol. II. page 299.)

Ex. 18. Ans. $\frac{2}{3}x^{-\frac{1}{3}}\dot{x} = \frac{\frac{2}{3}\dot{x}}{\sqrt[3]{x}} = \frac{\dot{x}}{1\frac{1}{2}x^{\frac{1}{2}}}$.

Ex. 19. Ans. $\frac{3}{2}x^{\frac{1}{2}}\dot{x} = 1\frac{1}{2}\sqrt{x}\dot{x}$.

Ex. 20. Ans. $\frac{3}{4}x^{-\frac{1}{4}}\dot{x} = \frac{\frac{3}{4}\dot{x}}{\sqrt[4]{x}} = \frac{\dot{x}}{1\frac{1}{3}\sqrt[4]{x}}$.

Ex. 21. Ans. $1\frac{1}{3}x^{\frac{1}{3}}\dot{x} = \frac{4}{3}\sqrt[3]{x}\dot{x}$.

Ex. 22. Ans. $\frac{1}{2}(a^2+x^2)^{-\frac{1}{2}} \times 2x\dot{x} = \frac{x\dot{x}}{\sqrt{a^2+x^2}}$.

Ex. 23. Ans. $\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{\sqrt{a^2-x^2}}$.

Ex. 24. The fluxion of $2rx-x^2$ being $2r\dot{x}-2x\dot{x}$, or $2(r-x)\dot{x}$, it is $\frac{1}{2}(2rx-x^2)^{-\frac{1}{2}} \times 2(r-x)\dot{x}$, or

$$\frac{(r-x)\dot{x}}{\sqrt{2rx-x^2}}. \quad \text{Ans.}$$

Ex. 25. Ans. $-\frac{1}{2}(a^2-x^2)^{-\frac{3}{2}} \times -2x\dot{x} = \frac{x\dot{x}}{\sqrt{(a^2-x^2)^3}}$.

Ex. 26. The fluxion of $ax-x^2$ being $a\dot{x}-2x\dot{x}$, or $(a-2x)\dot{x}$, it is $\frac{1}{3}(ax-x^2)^{-\frac{2}{3}} \times (a-2x)\dot{x}$, or

$$\frac{(a-2x)\dot{x}}{3\sqrt[3]{\overline{ax-x^2}|^2}}. \quad \text{Ans.}$$

(Page 300.)

Ex. 27. Ans. $\frac{2(\pm x\dot{x})\dot{x}}{\sqrt{a^2 \pm x^2}} = \frac{\pm 2x\dot{x}^2}{(a^2 \pm x^2)^{\frac{1}{2}}}$.

Ex. 28. Ans. $\frac{3}{2}(a^2-x^2)^{\frac{1}{2}} \times -2x\dot{x} = -3x(a^2-x^2)^{\frac{1}{2}}\dot{x}$.

(Key to Vol. II. page 300.)

Ex. 29. Ans. $\frac{\dot{x}z+x\dot{z}}{2\sqrt{xz}}$.

Ex. 30. The fluxion of $xz-z^2$ being $\dot{x}z+x\dot{z}-2z\dot{z}$, or $(x-2z)\dot{z}+z\dot{x}$, it is $\frac{1}{2}(xz-z^2)^{-\frac{1}{2}}\times z\dot{x}+(x-2z)\dot{z}$, that is,

$$\frac{z\dot{x}+(x-2z)\dot{z}}{2\sqrt{xz-z^2}} \quad \text{Ans.}$$

Ex. 31. Ans. $\frac{1}{2a}\times x^{-\frac{3}{2}}\times -\dot{x} = \frac{-\dot{x}}{2ax^{\frac{3}{2}}}$.

Ex. 32. The fluxion of the numerator drawn into the denominator is $3ax^2\dot{x}\,(a+x)=3a^2x^2\dot{x}+3ax^3\dot{x}$. And the fluxion of the denominator drawn into the numerator is $ax^3\dot{x}$.

Hence $\frac{(3a^2x^2+2ax^3)\dot{x}}{a^2+2ax+x^2}$ Ans.

Ex. 33. Here the fluxion of the numerator drawn into the denominator is $mx^{m-1}y^n\dot{x}$. And the fluxion of the denominator drawn into the numerator is $nx^m y^{n-1}\dot{y}$.

Hence $\frac{mx^{m-1}y^n\dot{x}-nx^my^{n-1}\dot{y}}{y^{2n}}$. Ans.

Ex. 34. The fluxion of the numerator drawn into the denominator is $z(\dot{x}y+x\dot{y})$. And the fluxion of the denominator drawn into the numerator is $xy\dot{z}$.

Therefore $\frac{z(\dot{x}y+x\dot{y})-xy\dot{z}}{z^2}$. Ans.

Ex. 35. $\frac{c}{xx}=cx^{-2}$. Consequently $-2cx^{-3}\dot{x}$, or $\frac{-2c\dot{x}}{x^3}$. Ans.

Or, without the negative index;

The fluxion of the numerator is 0, which drawn into the denominator is 0; and the fluxion of the denominator is $2x\dot{x}$, which drawn into the numerator is $2cx\dot{x}$: this subtracted from 0 leaves $-2cx\dot{x}$: lastly $\frac{-2cx\dot{x}}{x^4}=\frac{-2c\dot{x}}{x^3}$ *as before.*

(Key to Vol. II. page 300.)

Ex. 36. In this Example the rectangle under the denominator and fluxion of the numerator is $3(a-x)\dot{x} = 3a\dot{x} - 3x\dot{x}$; and the rectangle under the numerator and fluxion of the denominator is $3x\dot{x}$. Wherefore $\frac{(3a-6x)\dot{x}}{(a-x)^2}$. Ans.

Ex. 37. The rectangle under the denominator and fluxion of the numerator is $(x+z)\dot{z} = x\dot{z} + z\dot{z}$; and the rectangle under the numerator and fluxion of the denominator is $z(\dot{x}+\dot{z}) = z\dot{x} + z\dot{z}$.

Therefore $\frac{\dot{z}x - z\dot{x}}{(x+z)^2}$. Ans.

Ex. 38. Here the fluxion of the numerator drawn into the denominator is $2xz^2\dot{x}$; and the fluxion of the denominator drawn into the numerator is $2x^2z\dot{z}$. Consequently

$$\frac{2(xz^2\dot{x} - x^2z\dot{z})}{z^4}, \text{ or } \frac{xz\dot{x} - x^2\dot{z}}{\frac{1}{2}z^3}. \quad \text{Ans.}$$

Ex. 39. The fluxion of $x^{\frac{2}{3}}$ is $\frac{2}{3}x^{-\frac{1}{3}}\dot{x}$. And the fluxion of $y^{\frac{3}{2}}$ is $\frac{3}{2}\sqrt{y}\dot{y}$.

Wherefore $\frac{\frac{2}{3}y^{\frac{3}{2}}x^{-\frac{1}{3}}\dot{x} - \frac{3}{2}x^{\frac{2}{3}}y^{\frac{1}{2}}\dot{y}}{y^3}$. Ans.

Otherwise.

$$\frac{x^{\frac{2}{3}}}{y^{\frac{3}{2}}} = x^{\frac{2}{3}}y^{-\frac{3}{2}} \text{ of which the fluxion is}$$

$$\frac{2}{3}x^{-\frac{1}{3}}y^{-\frac{3}{2}}\dot{x} - \frac{3}{2}x^{\frac{2}{3}}y^{-\frac{5}{2}}\dot{y} = \frac{\frac{2}{3}\dot{x}}{x^{\frac{1}{3}}y^{\frac{3}{2}}} - \frac{\frac{3}{2}x^{\frac{2}{3}}\dot{y}}{y^{\frac{5}{2}}} = \frac{\frac{2}{3}y^{\frac{5}{2}}\dot{x} - \frac{3}{2}xy^{\frac{3}{2}}\dot{y}}{x^{\frac{1}{3}}y^4} =$$

$$\frac{\frac{2}{3}y^{\frac{3}{2}}\dot{x} - \frac{3}{2}xy^{\frac{1}{2}}\dot{y}}{x^{\frac{1}{3}}y^3} = \frac{\frac{2}{3}y^{\frac{3}{2}}x^{-\frac{1}{3}}\dot{x} - \frac{3}{2}x^{\frac{2}{3}}y^{\frac{1}{2}}\dot{y}}{y^3} \text{ *as before.*}$$

Ex. 40. The fluxion of axy^2 is $a\dot{x}y^2 + 2axy\dot{y}$; and the fluxion of z is $\dot{z}$.

Therefore $\frac{az(\dot{x}y^2 + 2xy\dot{y}) - axy^2\dot{z}}{z^2}$. Ans.

(Key to Vol. II. page 300.)

Ex. 41. $\frac{3}{\sqrt{(x^2-y^2)}}=3(x^2-y^2)^{-\frac{1}{2}}$; and the fluxion of x^2-y^2 is $2x\dot{x}-2y\dot{y}$. Therefore $-\frac{3}{2}(x^2-y^2)^{-\frac{3}{2}}\times 2(x\dot{x}-y\dot{y})=\frac{-3(x\dot{x}-y\dot{y})}{(x^2-y^2)^{\frac{3}{2}}}$. Ans.

Otherwise.

The fluxion of 3 is 0; and the fluxion of $\sqrt{x^2-y^2}$ being $\frac{1}{2}(x^2-y^2)^{-\frac{1}{2}}\times 2x\dot{x}-2y\dot{y}$, the rectangle under the numerator and fluxion of the denominator is $\frac{3(x\dot{x}-y\dot{y})}{(x^2-y^2)^{\frac{1}{2}}}$; this subtracted from 0 leaves $\frac{-3(x\dot{x}-y\dot{y})}{(x^2-y^2)^{\frac{1}{2}}}$, which, divided by the square of $\sqrt{x^2-y^2}=x^2-y^2$, becomes $\frac{-3(x\dot{x}-y\dot{y})}{(x^2-y^2)^{\frac{3}{2}}}$. *as before.*

Ex. 42. Here the fluxion of the quantity is $a\dot{x}$, which divided by the quantity becomes $\frac{a\dot{x}}{ax}=\frac{\dot{x}}{x}$. Ans.

Ex. 43. In this Example the fluxion of the quantity is $\dot{x}$, which divided by the quantity becomes $\frac{\dot{x}}{1+x}$. Ans.

Ex. 44. Ans. $\frac{\dot{x}}{1-x}$.

Ex. 45. Ans. $\frac{2x\dot{x}}{x^2}=\frac{2\dot{x}}{x}=\frac{\dot{x}}{\frac{1}{2}x}$.

Ex. 46. Ans. $\frac{\frac{1}{2}z^{-\frac{1}{2}}\dot{z}}{z^{\frac{1}{2}}}=\frac{\frac{1}{2}\dot{z}}{z}=\frac{\dot{z}}{2z}$.

Ex. 47. Ans. $\frac{mx^{m-1}\dot{x}}{x^m}=\frac{m\dot{x}}{x}=\frac{\dot{x}}{\frac{1}{m}x}$

(Key to Vol. II. page 301.)

Ex. 48. $\frac{2}{x^2} = 2x^{-2}$, of which the fluxion is $-4x^{-3}\dot{x} = \frac{-4\dot{x}}{x^3}$, and this fluxion divided by $\frac{2}{x^2} = \frac{-2\dot{x}}{x}$. Ans.

Ex. 49. The fluxion of $\frac{1+x}{1-x}$ is $\frac{-2x\dot{x}}{(1-x)^2}$, which divided by $\frac{1+x}{1-x}$ becomes $\frac{-2x\dot{x}}{1-x^2}$. Ans.

Ex. 50. The fluxion of $\frac{1-x}{1+x}$ is $\frac{2x\dot{x}}{(1+x)^2}$, which divided by $\frac{1-x}{1+x}$ becomes $\frac{2x\dot{x}}{1-x^2}$. Ans.

Ex. 51, Ans: $c^x\dot{x} \times$ hyp. log. of c. [*See Art.* 25, *page* 295, *vol.* ii. *of the Mathematics.*]

Ex. 52. Ans. $10^x\dot{x} \times$ (2·3025850929940456840179914 &c. the hyperbolic logarithm of 10.)

Ex. 53. Ans. $(a+c)^x\dot{x} \times$ hyp. log. of $(a+c)$.

Ex. 54. Ans. $100^{xy} \times (\dot{x}y+x\dot{y}) \times$ (4·605170185988091368 &c. the hyperbolic logarithm of 100.)

Ex. 55. The fluxion of the given quantity considering *only* the root variable is $zx^{z-1}\dot{x}$, and the fluxion of x^z considering *only* the exponent variable is $x^z\dot{z} \times$ hyp. log. of x. The sum of these two fluxions is

$$zx^{z-1}\dot{x}+x^z\dot{z} \times \text{hyp. log. of } x. \quad \text{Ans.}$$

Ex. 56. The fluxion of y^{10x} considering the exponent constant is $10xy^{10x-1}\dot{y}$, and the fluxion of y^{10x} regarding the root constant is $10y^{10x}\dot{x} \times$ hyp. log. of y; wherefore $10xy^{10x-1}\dot{y} + 10y^{10x}\dot{x} \times$ hyp. log. of y. Ans.

Ex. 57. Ans. $x^x\dot{x}+x^x\dot{x} \times$ hyp. log. of x, or
(1+hyp. log. of x) $x^x\dot{x}$.

Ex. 58. The fluxion of $(xy)^{xz}$, the exponent considered constant, is $xz(xy)^{xz-1} \times (\dot{x}y+x\dot{y})$, and the fluxion of $(xy)^{xz}$, considering the root constant, is $(xy)^{xz} \times (\dot{x}z+x\dot{z}) \times$ hyp. log. of (xy). Therefore $xz(xy)^{xz-1} \times (\dot{x}y+x\dot{y})+(xy)^{xz} \times (\dot{x}z+x\dot{z}) \times$ hyp. log. of (xy). Ans.

(Key to Vol. II. page 301.)

Ex. 59. Ans. $\dot{x}\dot{y}\pm x\ddot{y}$.

Ex. 60. Ans. $\ddot{x}\dot{y}^2\pm 2\dot{x}\dot{y}^3$.

For $\dot{x}\dot{y}^2=\dot{x}\dot{y}\dot{y}$, of which the fluxion is $\ddot{x}\dot{y}\dot{y}+\dot{x}\ddot{y}\dot{y}+\dot{x}\dot{y}\ddot{y}=\ddot{x}\dot{y}^2+2\dot{x}\dot{y}^3$.

Ex. 61. The first fluxion of xy is $\dot{x}y+x\dot{y}$, of which the fluxion is $\ddot{x}y+\dot{x}\dot{y}+\dot{x}\dot{y}+x\ddot{y}=\ddot{x}y+2\dot{x}\dot{y}+x\ddot{y}$. Ans.

Ex. 62. The first fluxion of xy being as in the last Example $\dot{x}y+x\dot{y}$, it will be $\dot{x}\dot{y}+\dot{x}\dot{y}+x\ddot{y}=2\dot{x}\dot{y}+x\ddot{y}$. Ans.

Ex. 63. The first fluxion of x^n is $nx^{n-1}\dot{x}$, and the fluxion of this fluxion is $(n^2-n)x^{n-2}\dot{x}^2+nx^{n-1}\ddot{x}$. Ans.

Ex. 64. The first fluxion of x^n being, as in the last Example, $nx^{n-1}\dot{x}$, it will be $(n^2-n)x^{n-2}\dot{x}\times\dot{x}$ for the second fluxion, and $(n^3-3n^2+2n)x^{n-3}\dot{x}^3$ for the third fluxion, $\dot{x}$ being constant. Ans.

Ex. 65. The first fluxion of xy is $\dot{x}y+x\dot{y}$, of which the fluxion is $\ddot{x}y+2\dot{x}\dot{y}+x\ddot{y}$; and the fluxion of this fluxion is $\dddot{x}y+\ddot{x}\dot{y}+2\ddot{x}\dot{y}+2\dot{x}\ddot{y}+\dot{x}\ddot{y}+x\dddot{y}=\dddot{x}y+3\ddot{x}\dot{y}+3\dot{x}\ddot{y}+x\dddot{y}$. Ans.

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Ex. 8. Ans. y^n.

Ex. 9. Ans. $\frac{-1}{z}$.

Ex. 10. $\frac{a\dot{y}}{y^n}=ay^{-n}\dot{y}$, of which the fluent is $\frac{\frac{1}{1-n}a}{y^{n-1}}$. Ans.

Ex. 11. Ans. $\frac{1}{5}(a+x)^5$.

Ex. 12. Ans. $\frac{1}{8}(a^4+y^4)^2$.

Ex. 13. Ans. $\frac{1}{15}(a^3+z^3)^5$.

Ex. 14. Ans. $\frac{1}{mn+n}(a^n+x^n)^{m+1}$.

(Key to Vol. II. page 303.)

Ex. 15. Ans. $\frac{1}{8}(a^2+y^2)^4$.

Ex. 16. Ans. $(a^2+z^2)^{\frac{1}{2}}$.

Ex. 17. Ans. $-2\sqrt{a-x}$.

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When a Compound Root is multiplied by some power of the flowing quantity such that, of this factor the Index plus 1 is a multiple of the Index of the unknown quantity under the vinculum.

Ex. 1. Put z for $a+cx$, so shall $x=\frac{z-a}{c}$; and taking the fluxion on both sides, $\dot{x}=\frac{\dot{z}}{c}$. Consequently $x^3\dot{x} = (\frac{z-a}{c})^3 \times \frac{\dot{z}}{c} = \frac{z^3 - 3az^2 + 3a^2z - a^3}{c^4}\dot{z}$. Hence $(a+cx)^{\frac{1}{2}}x^3\dot{x} = z^{\frac{1}{2}} \times \frac{z^3 - 3az^2 + 3a^2z - a^3}{c^4}\dot{z}$, an expression equivalent to $\frac{z^{\frac{7}{2}} - 3az^{\frac{5}{2}} + 3a^2z^{\frac{3}{2}} - a^3}{c^4}\dot{z}$, which is equal to $\frac{z^{\frac{7}{2}}\dot{z}}{c^4} - \frac{3az^{\frac{5}{2}}\dot{z}}{c^4} + \frac{3a^2z^{\frac{3}{2}}\dot{z}}{c^4} - \frac{a^3\dot{z}}{c^4}$, of which the fluent is $\frac{2z^{\frac{9}{2}}}{9c^4} - \frac{6az^{\frac{7}{2}}}{7c^4} + \frac{6a^2z^{\frac{5}{2}}}{5c^4} - \frac{a^3z}{c^4}$.

Now substituting $a+cx$ for z, this fluent becomes

$$\frac{\frac{2}{9}(a+cx)^{\frac{9}{2}} - \frac{6}{7}a(a+cx)^{\frac{7}{2}} + \frac{6}{5}a^2(a+cx)^{\frac{5}{2}} - a^3(a+cx)}{c^4}. \text{ Ans.}$$

Ex. 2. Assuming $z = a+cx$ as in the last Example, $x^2\dot{x} = (\frac{z-a}{c})^2 \times \frac{\dot{z}}{c} = \frac{z^2 - 2az + a^2}{c^3}\dot{z}$. Also $(a+cx)^{\frac{3}{4}}x^2\dot{x} = z^{\frac{3}{4}} \times \frac{z^2 - 2az + a^2}{c^3}\dot{z}$, an expression equivalent to $\frac{z^{\frac{11}{4}} - 2az^{\frac{7}{4}} + a^2z^{\frac{3}{4}}}{c^3}\dot{z}$,

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and equal to $\frac{z^{\frac{11}{4}}\dot{z}-2az^{\frac{7}{4}}\dot{z}+a^2z^{\frac{3}{4}}\dot{z}}{c^3}$; now of this fluxion the fluent is $\frac{\frac{4}{13}z^{\frac{13}{4}}-\frac{8}{9}az^{\frac{9}{4}}+\frac{4}{5}a^2z^{\frac{5}{4}}}{c^3}$. Substituting therefore $a+cx$ for z, it is $\frac{\frac{4}{13}(a+cx)^{\frac{13}{4}}-\frac{8}{9}(a+cx)^{\frac{9}{4}}+\frac{4}{5}a^2(a+cx)^{\frac{5}{4}}}{c^3}$. Ans.

Ex. 3. Let y represent $a+cx^2$, then $x^2=\frac{y-a}{c}$. But the fluxion of this equation is $2x\dot{x}=\frac{\dot{y}}{c}$; or by dividing both sides by 2, $x\dot{x}=\frac{\dot{y}}{2c}$: therefore (multiplying the first side by x^2, and the second by its equivalent,) $x^3\dot{x}=(\frac{y-a}{c^2})\dot{y}$. Consequently $d\,(a+cx^2)^{\frac{1}{3}}=dy^{\frac{1}{3}}\times(\frac{y-a}{c^2})\dot{y}=\frac{dy^{\frac{4}{3}}\dot{y}-ady^{\frac{1}{3}}\dot{y}}{c^2}$. Now of this fluxion the fluent is $\frac{\frac{3}{7}dy^{\frac{7}{3}}-\frac{3}{4}ady^{\frac{4}{3}}}{c^2}$, or, (substituting $a+cx^2$, for y,) $\frac{\frac{3}{7}d\,(a+cx^2)^{\frac{7}{3}}-\frac{3}{4}ad(a+cx^2)^{\frac{4}{3}}}{c^2}$. Ans.

Ex. 4. If x be assumed for $a+z$, and their fluxions put equal, it will be $\dot{x}=\dot{z}$; therefore $z\dot{z}=x\dot{x}$. Consequently $c(a+z)^{-\frac{1}{2}}z\dot{z}=cx^{-\frac{1}{2}}x\dot{x}=cx^{\frac{1}{2}}\dot{x}$, of which the fluent is $\frac{2}{3}cx^{\frac{3}{2}}$. This, by substituting $a+z$ for x, becomes $\frac{2}{3}c(a+z)^{\frac{3}{2}}$. Ans.*

* *The fluent arising from a given fluxion often requires (as in the present Example) a correction to make it cotemporaneous with the fluxion.* [*See page* 313, *vol.* ii.]

Ex. 5. Assuming $x=a+z^n$, and taking the fluxion of the equation, $\dot{x}=nz^{n-1}\dot{z}$; also $x-a=z^n$, and by squaring both sides, $x^2-2ax+a^2=z^{2n}$. Consequently $(x^2-2ax+a^2)\dot{x}=nz^{3n-1}\dot{z}$, and $(a+z^n)^{-\frac{1}{2}}cz^{3n-1}\dot{z}=x^{-\frac{1}{2}}\times(\frac{x^2-2ax+a^2}{n})$

$$\times\,c\dot{x}=\frac{cx^2\dot{x}-2acx\dot{x}+a^2c\dot{x}}{n\sqrt{x}}=\frac{cx^{\frac{3}{2}}\dot{x}-2acx^{\frac{1}{2}}\dot{x}+a^2cx^{-\frac{1}{2}}\dot{x}}{n},$$

(Key to Vol. II. page 304.)

of which the fluent is $\frac{\frac{2}{5}cx^{\frac{5}{2}}-\frac{4}{3}acx^{\frac{3}{2}}+2a^2cx^{\frac{1}{2}}}{n}$. Substituting therefore $a+z^n$ for x, it is $\frac{c}{n}\times\left\{\frac{2}{5}(a+z^n)^{\frac{5}{2}}-\frac{4}{3}a(a+z^n)^{\frac{3}{2}}+2a^2(a+z^n)^{\frac{1}{2}}\right\}$ Ans.

Ex. 6. Ans. $\frac{\overline{a^2+z^2}|^{\frac{3}{2}}\times\overline{2z^2-3a^2}}{15a^4z^5}$.

Ex. 7. Ans. $\frac{\overline{a-x^n}|^{\frac{3}{2}}}{-\frac{7}{2}nax^{\frac{7}{2}n}}\times\overline{1+\frac{4x^n}{5a}+\frac{8x^{2n}}{15a^2}}|$ which reduced is

$$\frac{\overline{a-x^n}|^{\frac{3}{2}}\times\overline{30a^2+24ax^n+16x^{2n}}}{105na^3x^{\frac{7}{2}n}}.$$

When there are several Terms involving Two or more variable Quantities.

Ex. 3. Ans. $x^{\frac{1}{2}}y^2$.

Ex. 4. Ans. $\frac{x}{y}$.

Ex. 5. Ans. $ax^2y^{-\frac{1}{2}}$.

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When the Fluxion of a Quantity is divided by the Quantity.

Ex. 4. Ans. The hyp. log. of $a+x$.

Ex. 5. Ans. The hyp. log. of $a+x^3$.

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The Comparison of Fluents.

Ex. 4. The proposed fluxion is found to agree with the 1*st* FORM; and $x^4\dot{x}=x^{n-1}\dot{x}$; hence $n=5$.

Therefore $\frac{ax^5}{5}$, or $\frac{a}{5}x^5$. Ans.

(Key to Vol. II. page 309.)

Ex. 5. This fluxion corresponds with the 2*nd* Form,
Wherein $x^2 = x^n$, and $n = 2$.
Also $m - 1 = \frac{2}{3}$, or $m = \frac{5}{3}$.
Therefore $2 \times \frac{3}{10}(10 + x^2)^{\frac{5}{3}}$, or
$\frac{3}{5}(10 + x^2)^{\frac{5}{3}}$. Ans.

Ex. 6. There being numerous typographical errors in the fluxional part of Hutton's Course, it is imagined the present Example was misprinted. If the fluent of $\frac{a\dot{x}}{(c^2 + x^2)^{\frac{2}{2}}}$ be really required, it is $\frac{a}{c} \times$ arc to tang. $\frac{1}{c}\sqrt{x}$, or $\frac{a}{2c} \times$ arc to cos. $\frac{c^2 - x^2}{c^2 + x^2}$; *according to the* 11*th* Form.*

Ex. 7. In this Example the fluxion agrees with the 3*d* Form, and $x^2 = x^{mn-1}$; hence $m = 3$, and $n = 1$.

Consequently $\frac{3}{1} \times \frac{1}{3}a \times \frac{x^3}{(a-x)^3}$, or $\frac{ax^3}{(a-x)^3}$. Ans.

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Ex. 8. Here the fluxion is of the 4*th* Form, also $m = n = 2$; wherefore the fluent of $\frac{(c^2 - x^2)^1 \dot{x}}{x^5}$ is $\frac{-1}{4c^2} \times \frac{(c^2 - x^2)^2}{x^4} =$ $\frac{-c^2}{4x^4} + \frac{1}{2x^2} - \frac{1}{4}c^2$. Ans.

Ex. 9. Ans. $\frac{-1}{6x^3} - \frac{3}{4x^2}$; *by the* 4*th* Form.

Ex. 10. Ans. x^3y^2; *by the* 5*th* Form.

Ex. 11. Ans. $xy^{\frac{1}{3}}$; *by the* 5*th* Form.

Ex. 12. Ans. $\frac{3}{a} \times$ hyp. log. of x; *by the* 7*th* Form.

Ex. 13. Ans. $-\frac{1}{2}a$† × hyp. log. of $3 - 2x$; *by the* 8*th* Form.

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* *In the* 11*th Form tang.* $\sqrt{\frac{x}{a}}$ *should be tang.* $\sqrt{\frac{x^n}{a}}$, *and consequently the Answer to this Example, as well as to Examples* 18 *and* 19, *and to all others by the* 11*th Form are in part incorrect.* See Key, page 295.

† Because the fluent of $\frac{a\dot{x}}{3 - 2x} = -\frac{1}{2}a \times$ fluent of $\frac{-2\dot{x}}{3 - 2x}$.

(Key to Vol. II. page 310.)

Ex. 14.* Ans. $\frac{3}{2}$ hyp. log. of $\frac{x}{2-x}$, *by the* 9*th* Form, which is likewise misprinted, and ought to be $\frac{x^{-1}\dot{x}}{a \pm x^n}$.

* *There is a typographical error in the denominator of this Example in every edition of the* Course.

Ex. 15.† The given Example, when corrected, is equal to $\frac{2}{3} \times \frac{x^{-1}\dot{x}}{\frac{1}{3} - x^2}$, a Form agreeing with the 9*th* in the Table; therefore the hyp. log. of $\frac{x^2}{\frac{1}{3} - x^2}$. Ans.

† The latter part of this Example is misprinted in the Mathematics. It ought to be $\frac{2x^{-1}\dot{x}}{1-3x^2}$ instead of $\frac{2x^{-1}3\dot{z}^2}{1-3\dot{x}^2}$.

Ex. 16. Ans. $\frac{3}{4}$ hyp. log. of $\frac{1+x^2}{1-x^2}$, *by the* 10*th* Form,† a being 1 and $n = 4$.

† Which in every copy we have seen of **Hutton** is misprinted. It ought to be $\frac{x^{\frac{1}{2}n-1}}{a-x^n}$

Ex. 17. Ans. $\frac{a}{1} \times \frac{1}{5\sqrt{2}} \times$ hyp. log. of $\frac{\sqrt{2}+\sqrt{x^5}}{\sqrt{2}-\sqrt{x^5}}$, *by the* 10*th* Form, n being 5, and a in the form $= 2$.

Ex. 18. Ans. $\frac{2}{1} \times \frac{2}{4\sqrt{1}} \times$ arc to tan. $\sqrt{\frac{x^{\ddagger}}{1}}$
Or, $\frac{2}{1} \times \frac{1}{4\sqrt{1}} \times$ arc to cos. $\frac{1-x^4}{1+x^4}$ } *by the* 11*th* Form.

That is, The arc to the tangent of $x^{\frac{1}{2}}$;
or, half the arc to the cosine of $\frac{1-x^4}{1+x^4}$. } *by reduction.*

‡ *See the note in page* 295 *of the* Key.

(Key to Vol. II. page 310.)

$$\left.\begin{array}{l}\text{Ex. 19. Ans. } \frac{a}{1} \times \frac{2}{5\sqrt{2}} \times \text{arc to tang. } \sqrt{\frac{x}{2}};^{*} \\ \text{or, } \frac{a}{1} \times \frac{1}{5\sqrt{2}} \times \text{arc to cos. } \frac{2-x^5}{2+x^5}.\end{array}\right\} \textit{by } \text{FORM 11.}$$

$$\left.\begin{array}{l}\text{And, } \tfrac{1}{5}\, a\sqrt{2} \times \text{arc to tang. } \tfrac{1}{2}\sqrt{2x}; \\ \text{or, } \tfrac{1}{10}\, a\sqrt{2} \times \text{arc to cos. } \frac{2-x^5}{2+x^5}\end{array}\right\} \textit{by reduction.}$$

* See the note referred to in the last Example.

Ex. 20. Ans. $\frac{6}{4} \times$ hyp. log. of $\sqrt{x^4}+\sqrt{1+x^4}$, *by the* 12*th* FORM. and, $\frac{3}{2}$ hyp. log. $x^2+\sqrt{1+x^4}$, *by reduction.*

Ex. 21. Changing the given expression to its equivalent $\frac{ax^{\circ}\dot{x}}{\sqrt{-4+x^2}} = \frac{a}{1} \times \frac{x^{\circ}\dot{x}}{\sqrt{-4+x^2}}$, the FORM agrees with the 12*th* in the Table, n being 2, and $a=-4$. Therefore $a \times$ hyp. log. of $x+\sqrt{x^2-4}$. Ans.

NOTE. *x° is always equivalent to unity, which may be proved thus;* $x^{\circ}=x^{1-1}=x^1 \times x^{-1}=\frac{x}{1} \times \frac{1}{x}=\frac{x}{x}=1$.

Ex. 22. Ans. $\frac{3}{2} \times$ arc to sin. x^2, or $\frac{3}{4} \times$ arc to vers. $2x^4$, *by the* 13*th* FORM n being 4 and $a=1$.

Ex. 23. Ans. $a \times$ arc to sin. $\frac{1}{2}x$, or $\frac{1}{2}a \times$ arc to vers. $\frac{1}{2}x^2$, *by the* 13*th* FORM, *a* in the FORM being 4 and $n=2$. *See Example* 21 *above, or Examples* 6 *and* 13, *page* 286.

Ex. 24. *By the* 14*th* FORM, n being 2, and $a=1$, the fluent is the hyp. log. of $\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = 2$ hyp. log. $\frac{1-\sqrt{1-x^2}}{x}$.

Ex. 25. $$\frac{a\dot{x}}{\sqrt{ax^2+x^{\frac{1}{2}}}} = \frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}}}\sqrt{ax^{\frac{3}{2}}+1}} = \frac{ax^{-\frac{1}{4}}\dot{x}}{\sqrt{ax^{\frac{3}{2}}+1}} = \frac{ax^{-\frac{1}{4}}\dot{x}}{\sqrt{a\sqrt{x^{\frac{3}{2}}+a^{-1}}}}$$

(Key to Vol. II. page 310.)

$$=\frac{a^{\frac{1}{2}}x^{-\frac{1}{4}}\dot{x}}{\sqrt{\frac{1}{a}+x^{\frac{3}{2}}}}=\sqrt{a}\times\frac{x^{-\frac{1}{4}}\dot{x}}{\sqrt{a^{-1}+x^{\frac{3}{2}}}},$$ a form agreeing with the 12*th* in the Table, n being $\frac{3}{2}$, and a under the vinculum in the Table $=\frac{1}{a}$ in the proposed Example; therefore

$$\tfrac{4}{3}\sqrt{a}\times\text{hyp. log. of }\sqrt{x^{\frac{3}{2}}}+\sqrt{a^{-1}+x^{\frac{3}{2}}}.\quad\text{Ans.}$$

Otherwise,

By the 14*th* FORM. The given fluxion divided by x becomes $\frac{ax^{-1}\dot{x}}{\sqrt{a+x^{-\frac{3}{2}}}}$, of which the fluent is $-\frac{2}{3}\sqrt{a}\times\text{hyp. log.}\,\frac{\sqrt{a+x^{-\frac{3}{2}}}-\sqrt{a}}{\sqrt{a+x^{-\frac{3}{2}}}+\sqrt{a}}$, *reducible to simpler terms.*

Ex. 26. Correcting the typographical error in the denominator, &c. and dividing the numerator by 2, the fluxion becomes $\frac{x^{-1}\dot{x}}{\sqrt{x^2-1}}$, agreeing with the 15*th* FORM. Therefore twice the arc to secant x, or the arc to cosine $\frac{2-x^2}{x^2}$. Ans.

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Ex. 27. $\frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}}-ax^2}}=\frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}}}\sqrt{-ax^{\frac{3}{2}}+1}}$ which is equal to $\sqrt{a}\times\frac{x^{-\frac{1}{4}}\dot{x}}{\sqrt{-x^{\frac{3}{2}}+a^{-1}}}$, a form agreeing with the 13*th* of the Table; therefore

$$\left.\begin{array}{r}\tfrac{4}{3}\sqrt{a}\times\text{arc to sin. }\sqrt{a^{-1}x^{\frac{3}{2}}},\\ \text{or, }\tfrac{2}{3}\sqrt{a}\times\text{arc to vers. }2a^{-1}x^{\frac{3}{2}}\end{array}\right\}\text{Ans.}$$

Ex. 28. Ans. Circular segment to diameter 2 and versed sine x, *by the* 16*th* FORM.

Ex. 29. Ans. $\frac{a^x}{\text{hyp. log. of }a}$, *by the* 17*th* FORM, n being 1.

Ex. 30. Ans. $\frac{3a^{2x}}{\text{twice hyp. log. of }a}$, *by the* 17*th* FORM, n being 2.

Ex. 31. Ans. $3z^x$, *by the* 18*th* FORM.

(Key to Vol. II. page 311.)

Ex. 32. $(1+x^3)x\dot{x}=(x+x^4)\dot{x}=x\dot{x}+x^4\dot{x}$, of which the fluent *by the* 1*st* FORM is $\frac{1}{2}x^2+\frac{1}{5}x^5$. Or if the given fluxion be $(1+x^3)x^2\dot{x}$ the fluent is $\frac{1}{3}x^3+\frac{1}{6}x^6$. Ans.

Ex. 33. $(2+x^4)x^{\frac{3}{2}}\dot{x}=(2x^{\frac{3}{2}}+x^{\frac{11}{2}})\dot{x}=2x^{\frac{3}{2}}\dot{x}+x^{\frac{11}{2}}\dot{x}$, of which the fluent *by the* 1*st* FORM is $\frac{4}{5}x^{\frac{5}{2}}+\frac{2}{13}x^{\frac{13}{2}}$. Ans.

Ex. 34. $x^2\dot{x}\sqrt{a^2+x^2}=\dfrac{x^2\dot{x}(a^2+x^2)}{\sqrt{a^2+x^2}}=\dfrac{a^2x^2\dot{x}}{\sqrt{a^2+x^2}}+\dfrac{x^4\dot{x}}{\sqrt{a^2+x^2}}$.

Multiplying both numerator and denominator of the first part by x, and of the second by x^3, the result is $\dfrac{a^2x^3\dot{x}}{\sqrt{a^2x^2+x^4}}+\dfrac{x^7\dot{x}}{\sqrt{a^2x^6+x^8}}=x^2\dot{x}\sqrt{a^2+x^2}$.

Thus transformed, the quantity without the vinculum is a constant part of the fluxion of the highest term under the vinculum; namely in the first fluxion, $\frac{1}{4}$th part; and in the second, $\frac{1}{8}$th part.

Adding therefore to the numerator in the one instance $\frac{1}{4}$th of, and in the other $\frac{1}{8}$th of, the fluxion of the leading term respectively under the vinculum, there arise

$a^2\times\dfrac{(\frac{1}{2}a^2x\dot{x}+x^3\dot{x})}{\sqrt{a^2x^2+x^4}}$ and $\dfrac{\frac{3}{4}a^2x^5\dot{x}+x^7\dot{x}}{\sqrt{a^2x^6+x^8}}$, two *perfect* fluxions.

Now the fluents of these by the 2*nd* FORM are $\frac{1}{2}a^2\sqrt{a^2x^2+x^4}$ and $\frac{1}{4}\sqrt{a^2x^6+x^8}$; the former equal to $\frac{1}{2}a^2x\sqrt{a^2+x^2}$, the latter equal to $\frac{1}{4}x^3\sqrt{a^2+x^2}$. But from the *former* there must be deducted the fluent of $\dfrac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2+x^2}}$ which by the 12*th* FORM is $\frac{1}{2}a^2\times$ hyp. log. of $x+\sqrt{a^2+x^2}$; and from the *latter*, the fluent of $\dfrac{\frac{3}{4}a^2x^5\dot{x}}{\sqrt{a^2x^6+x^8}}$, an expression equivalent to $\dfrac{\frac{3}{4}a^2x^2\dot{x}}{\sqrt{a^2+x^2}}$, and consequently its fluent *(see the fluent just found,)* $\frac{3}{4}a^2\times(\frac{1}{2}x\sqrt{a^2+x^2}-\frac{1}{2}a^2\times$ hyp. log. $\overline{x+\sqrt{a^2+x^2}}$.) Therefore $\frac{1}{2}a^2(x\sqrt{a^2+x^2}-$ hyp. log. of $\overline{x+\sqrt{a^2+x^2}})+\Big\{(\frac{1}{4}x^3\sqrt{a^2+x^2}-\frac{3}{8}a^2(x\sqrt{a^2+x^2}-a^2\times$

hyp. log. of $\overline{x+\sqrt{a^2+x^2}}$.) $\}$ Or $(\frac{1}{4}x^3+\frac{1}{8}a^2x)\times\sqrt{a^2+x^2}$ $-\frac{1}{8}a^4\times$ hyp. log. of $(x+\sqrt{a\ \ +x^2})$. Ans.

To find Fluents by Infinite Series.

Ex. 1. For the ascending series.

$a-x)\ bx\ (\frac{bx}{a}+\frac{bx^2}{a^2}+\frac{bx^3}{a^3}+\frac{bx^4}{a^4}+\frac{bx^5}{a^5}=\frac{b}{a}\times(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\&c.)$ Wherefore

$\frac{bx\dot{x}}{a-x}=\frac{b}{a}(x\dot{x}+\frac{x^2\dot{x}}{a}+\frac{x^3\dot{x}}{a^2}+\frac{x^4\dot{x}}{a^3}+\frac{x^5\dot{x}}{a^4}+\&c.)$ of which the fluent is $\frac{b}{a}(\frac{x^2}{2}+\frac{x^3}{3a}+\frac{x^4}{4a^2}+\frac{x^5}{5a^3}+\frac{x^6}{6a^4}+\&c.)$ Ans.

And for the descending series.

$-x+a)\ bx\ (-b-\frac{ab}{x}-\frac{a^2b}{x^2}-\frac{a^3b}{x^3}-\frac{a^4b}{x^4}-\frac{a^5b}{x^5}\ \&c.$

Therefore $\frac{bx\dot{x}}{a-x}=-b\dot{x}-\frac{ab\dot{x}}{x}-\frac{a^2b\dot{x}}{x^2}-\frac{a^3b\dot{x}}{x^3}-\frac{a^4b\dot{x}}{x^4}-\&c.$ of which the fluent is $-bx,-ab\times$ hyp. log. of $x,+a^2bx^{-1}+\frac{1}{2}a^3bx^{-2}+\frac{1}{3}a^4bx^{-3}+\&c.$ Ans.

Ex. 2. For the ascending series.

$a+x)\ b\ (\frac{b}{a}-\frac{bx}{a^2}-\frac{bx^2}{a^3}-\frac{bx^3}{a^4}-\&c.=\frac{b}{a}\times(1-\frac{x}{a}-\frac{x^2}{a^2}-\frac{x^3}{a^3}-\&c.)$

Wherefore $\frac{b\dot{x}}{a+x}=\frac{b}{a}(\dot{x}-\frac{x\dot{x}}{a}-\frac{x^2\dot{x}}{a^2}-\frac{x^3\dot{x}}{a^3}-\&c.)$ of which the fluent is $\frac{b}{a}\times(x-\frac{x^2}{2a}-\frac{x^3}{3a^2}-\frac{x^4}{4a^3}-\frac{x^5}{5a^4}-\&c.)$ Ans.

And for the descending series.

$x+a)\ b\ (\frac{b}{x}-\frac{ab}{x^2}-\frac{a^2b}{x^3}-\frac{a^3b}{x^4}-\frac{a^4b}{x^5}-\frac{a^5b}{x^6}-\&c.$ Hence $\frac{b\dot{x}}{a+x}=\frac{b\dot{x}}{x}-\frac{ab\dot{x}}{x^2}-\frac{a^2b\dot{x}}{x^3}-\frac{a^3b\dot{x}}{x^4}-\&c.$ of which the fluent is $b\times$ hyp. log. of $x,+abx^{-1}+\frac{1}{2}a^2bx^{-2}+\frac{1}{3}a^3bx^{-3}+\&c.$ Ans.

(Key to Vol. II. page 312.)

Ex. 3. $a+x)-3(-\frac{3}{a}+\frac{3}{a^2}x-\frac{3}{a^3}x^2+\frac{3}{a^4}x^3-\frac{3}{a^5}x^4+$ &c.

Therefore $-\frac{3}{a}\dot{x}+\frac{3}{a^2}x\dot{x}-\frac{3}{a^3}x^2\dot{x}+\frac{3}{a^4}x^3\dot{x}-$ &c.

And consequently $-\frac{3}{a}x+\frac{3}{2a^2}x^2-\frac{3}{3a^3}x^3+\frac{3}{4a^4}x^4-$ &c. Ans.

Ex. 4. $1+x-x^2)1-x^2+2x^4(1-x+x^2-2x^3+5x^4-7x^5+$ &c.

Wherefore $\dot{x}-x\dot{x}+x^2\dot{x}-2x^3\dot{x}+5x^4\dot{x}-7x^5\dot{x}+$ &c.

And $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{2}+x^5-\frac{7}{6}x^6+$ &c. Ans.

Ex. 5. $a^2+x^2)b(\frac{b}{a^2}-\frac{b}{a^4}x^2+\frac{b}{a^6}x^4-\frac{b}{a^8}x^6+\frac{b}{a^{10}}x^8-\frac{b}{a^{12}}x^{10}+$ &c.

Whence $\frac{b\dot{x}}{a^2}-\frac{bx^2\dot{x}}{a^4}+\frac{bx^4\dot{x}}{a^6}-\frac{bx^6\dot{x}}{a^8}+\frac{bx^8\dot{x}}{a^{10}}-\frac{b}{a^{12}}x^{10}\dot{x}+$ &c.

That is,

$z=\frac{bx}{a^2}-\frac{bx^3}{3a^4}+\frac{bx^5}{5a^6}-\frac{bx^7}{7a^8}+\frac{bx^9}{9a^{10}}-\frac{bx^{11}}{11a^{12}}+\frac{bx^{13}}{13a^{14}}-$ &c. Ans.

Ex. 6. $a+x)\ a^2+x^2\ (a-x+\frac{2x^2}{a}-\frac{2x^3}{a^2}+\frac{2x^4}{a^3}-\frac{2x^5}{a^4}+\frac{2x^6}{a^5}-$ &c.

And $a\dot{x}-x\dot{x}+\frac{2x^2\dot{x}}{a}-\frac{2x^3\dot{x}}{a^2}+\frac{2x^4\dot{x}}{a^3}-\frac{2x^5\dot{x}}{a^4}+$ &c. whence

$z=ax-\frac{1}{2}x^2+\frac{2x^3}{3a}-\frac{2x^4}{4a^2}+\frac{2x^5}{5a^3}-\frac{2x^6}{6a^4}+\frac{2x^7}{7a^5}-$ &c. Ans.

Ex. 7. $z=\frac{3}{2}(a+x)^2$. Ans.

Ex. 8. $\sqrt{a^2+x^2}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$ &c. therefore

$2a\dot{x}+\frac{x^2\dot{x}}{a}-\frac{x^4\dot{x}}{4a^3}+\frac{x^6}{8a^5}\dot{x}-$ &c $=\dot{z}$, and

$2ax+\frac{3x^3}{9a}-\frac{x^5}{20a^3}+\frac{x^7}{54a^5}-$ &c. $=z$. Ans.

Ex. 9. $\sqrt{a^2-x^2}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-$ &c. therefore

(Key to Vol. II. page 312.)

$$4a\dot{x} - \frac{2x^2\dot{x}}{a} - \frac{x^4\dot{x}}{2a^3} - \&\text{c.} = \dot{z}\text{; and,}$$

$$4ax - \frac{2x^3}{3a} - \frac{x^5}{10a^3} - \&\text{c. }\&\text{c.} = z. \quad \text{Ans.}$$

Ex. 10. $\frac{5a}{\sqrt{x^2-a^2}} = \frac{5a}{x} - \frac{10x}{a} - \frac{40x^3}{a^3} - \frac{80x^5}{a^5} - \&\text{c.}$ whence

$$5ax^{-1}\dot{x} - \frac{10x\dot{x}}{a} - \frac{40x^3\dot{x}}{a^3} - \frac{80x^5\dot{x}}{a^5} - \&\text{c.} = \dot{z}. \quad \text{And}$$

$$5a \times \text{hyp. log. of } x - \frac{5x^2}{a} - \frac{10x^4}{a^3} - \frac{80x^6}{6a^5} - \&\text{c.} = z. \text{ Ans.}$$

Ex. 11. $\sqrt[3]{a^3-x^3} = a - \frac{x^3}{3a^2} - \frac{x^6}{9a^5} - \frac{5x^9}{81a^8} - \&\text{c.}$ wherefore

$$2a\dot{x} - \frac{2x^3\dot{x}}{3a^2} - \frac{2x^6\dot{x}}{9a^5} - \&\text{c.} = \dot{z}\text{; consequently}$$

$$2ax - \frac{x^4}{6a^2} - \frac{2x^7}{63a^5} - \&\text{c.} = z. \quad \text{Ans.}$$

Ex. 12. $\frac{3a\dot{x}}{\sqrt{ax-xx}} = \frac{3\dot{x}}{1} \times \frac{a}{\sqrt{ax-xx}}.$ Likewise

$$\frac{a}{\sqrt{ax-xx}} = \frac{\sqrt{a}}{\sqrt{x}} - \frac{2a^{\frac{3}{2}}}{x^{\frac{3}{2}}} - \frac{8a^{\frac{5}{2}}}{x^{\frac{5}{2}}} - \frac{16a^{\frac{7}{2}}}{x^{\frac{7}{2}}} - \&\text{c.}$$

Hence, ×lying by $3\dot{x}$, it will be $3\sqrt{a}x^{-\frac{1}{2}}\dot{x} - 6a^{\frac{3}{2}}x^{-\frac{3}{2}}\dot{x} - 24a^{\frac{5}{2}}x^{-\frac{5}{2}}\dot{x} - 48a^{\frac{7}{2}}x^{-\frac{7}{2}}\dot{x} - \&\text{c.} = \dot{z}.$

Therefore $6\sqrt{a}x^{\frac{1}{2}} + 12a^{\frac{3}{2}}x^{-\frac{1}{2}} + 16a^{\frac{5}{2}}x^{-\frac{3}{2}} + 19\tfrac{1}{5}a^{\frac{7}{2}}x^{-\frac{5}{2}} + \&\text{c.} = z.$ Ans.

Ex. 13. $\sqrt[3]{x^3+x^4+x^5} = x + \frac{x^2}{3} + \frac{2x^3}{9} + \&\text{c.}$ therefore

$$2x\dot{x} + \frac{2x^2\dot{x}}{3} + \frac{4x^3\dot{x}}{9} + \&\text{c.} = \dot{z}\text{; and}$$

$$x^2 + \frac{2x^3}{9} + \frac{x^4}{9} + \&\text{c.} = z. \quad \text{Ans.}$$

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Ex. 14. $\sqrt{ax-x^2} = a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}} -$ &c. wherefore $5a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} - \frac{5x^{\frac{3}{2}}\dot{x}}{2\sqrt{a}} - \frac{5x^{\frac{5}{2}}\dot{x}}{8\sqrt{a^3}} - \frac{5x^{\frac{7}{2}}\dot{x}}{16\sqrt{a^5}} -$ &c. $= \dot{z}$; and consequently $\frac{10a^{\frac{1}{2}}x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{\sqrt{a}} - \frac{5x^{\frac{7}{2}}}{28\sqrt{a^3}} -$ &c. Ans.

(Page 314.)

The Correction of Fluents.

Ex. 2. Because $\dot{z}=5x\dot{x}$, it follows that $z=\frac{5}{2}x^2$. Putting, therefore, $z=0$, and $a=x$, it is $0=\frac{5}{2}a^2$, and by subtraction $z-0 = \frac{5x^2-5a^2}{2}$, that is, $z=\frac{5}{2}(x^2-a^2)$. Ans.

Otherwise,

z being unconditionally equal to $\frac{5}{2}x^2$, it will be conditionally $z=\frac{5}{2}x^2+c$, and by substituting 0 and a agreeably to the conditions, $c=-\frac{5}{2}a^2$; writing, therefore, this value of c for c, it is $z=\frac{5}{2}x^2-\frac{5}{2}a^2$, *as before.*

Ex. 3. Since $\dot{z}=3\dot{x}\sqrt{a+x}$, manifestly $z=2(a+x)^{\frac{3}{2}}$. Wherefore substituting 0 both for x and z, there arises $0=2(a+0)^{\frac{3}{2}}= 2\sqrt{a^3}$. Consequently the correct fluents are $z=2(a+x)^{\frac{3}{2}} -2\sqrt{a^3}$. Ans.

Or,

Putting c for the correction, $z=2(a+x)^{\frac{3}{2}}+c$; that is, (writing 0 both for z and x,) $0=2(a+0)^{\frac{3}{2}}+c$; hence $c=-2\sqrt{a^3}$, and the correct fluents are *as before* $z=2(a+x)^{\frac{3}{2}}-2\sqrt{a^3}$. Ans.

Ex. 4. Here $z=2a\times$ hyp. log. of $(a+x)$, and by substituting 0 both for x and z, agreeably to the conditions given, it is $z-0=2a\times$ hyp. log. of $(a+0)$, wherefore the correct fluents are $z=2a\times$ (hyp. log. of $(a+x)$ − hyp. log. of a) $=2a\times$ hyp. log. $\frac{(a+x)}{a}$. Ans.

(Key to Vol. II. page 314.)

Ex. 5. By the 11*th** Form $z=\frac{2}{a}\times$ arc to tang. $\frac{x^{\frac{1}{2}}}{a}$, or $\frac{1}{a}\times$ arc to cosine $\frac{a^2-x^2}{a^2+x^2}$. If, therefore, 0 be substituted both for x and z, there will arise in the *first* instance, $0=\frac{2}{a}\times$ arc to tang. of $0^{\frac{1}{2}}a^{-1}$, that is $\frac{2}{a}\times 0$; and in the *second* instance, $0=\frac{1}{a}\times$ arc to cos. $\frac{a^2-0}{a^2+0}=\frac{1}{a}\times$ arc to cos. 1. Consequently the correct fluents *according to the Table* are

$$z=\left\{\begin{array}{l}\frac{2}{a}\times \text{arc to tang. } x^{\frac{1}{2}}a^{-1}. \\ \frac{1}{a}\times (\text{arc to cos. } \frac{a^2-x^2}{a^2+x^2} - \text{arc to cos. } 1.)\end{array}\right\}\text{Ans.}$$

Or by Infinite Series.

$$\dot{z}=\frac{2\dot{x}}{a^2}\times\left(1-\frac{x^2}{a^2}+\frac{x^4}{a^4}-\frac{x^6}{a^6}+\frac{x^8}{a^8}-\frac{x^{10}}{a^{10}}+\&\text{c.}\right) \text{ and}$$

$$z=\frac{2x}{a^2}-\frac{2x^3}{3a^4}+\frac{2x^5}{5a^6}-\frac{2x^7}{7a^8}+\frac{2x^9}{9a^{10}}-\frac{2x^{11}}{11a^{12}}+\&\text{c.}$$

But substituting 0 both for x and z, it is $0=\frac{0}{a^2}-\frac{0}{3a^4}+\frac{0}{5a^6}-$ &c. signifying that $z=x-0$, or that the fluents require no correction.

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* We have taken the 11th form throughout as it is printed in Hutton; suggesting that the reader will easily accommodate our answers to any change in that form. Thus, when $\sqrt{xa^{-1}}$ is changed to $\sqrt{x^{n}a^{-1}}$, as it ought to be, will $x^{\frac{1}{2}}$ in the 1st and 7th lines of this page become x, and examples 6, 18, and 19, in the comparison of fluents be accordingly adjusted in the tangent value of each fluent. It is the business of the Key to shew how the fluents of proposed fluxions are found by the Table, not to construct or alter the Forms.

(Key to Vol. II. page 318.)

MAXIMA ET MINIMA.

Ex. 1. Put x for one of the parts of the given line or quantity, so shall $a-x$ be the other part. Now the rectangle under x and $a-x$ is $ax-x^2$, of which the fluxion is $a\dot{x}-2x\dot{x}$, by the question a maximum, and consequently $a-2x=0$. Hence $x=\frac{1}{2}a$. Ans.

Ex. 2. If x and y represent the two parts, it is evident that $\dot{x}=-\dot{y}$, and *vice versâ*. $\dot{y}=-\dot{x}$, according as x increases or decreases, the fluxion of both parts being simultaneous and equal. But by the question x^my^n is a maximum; therefore $mx^{m-1}y^n=ny^{n-1}x^m$; and, As $x^m:y^n::mx^{m-1}::ny^{n-1}::\frac{mx^m}{x}:\frac{ny^n}{y}::\frac{m}{x}:\frac{n}{y}$, consequently $my=nx$, and, As $x:y::m:n$. Ans.*

Ex. 3. Let x, y, and z be assumed for the three parts, then xyz by the question is a maximum: and because xyz is a maximum, xy is a maximum; and for the same reason xz is a maximum. But $\dot{x}=\dot{y}$, therefore, $x=y$; and $\dot{x}=\dot{z}$, consequently, $x=z$. Hence $x=y=z$. Ans.

Ex. 4. If x, y, z be assumed, xy^2z^3 is, by the question a maximum; And taking xy^2 a maximum $\dot{x}y^2-2xy\dot{y}=0$, hence $y=2x$: Again, taking xz^3 a maximum, $\dot{x}z^3-3xz^2\dot{z}=0$, consequently $z=3x$. Wherefore $x=\frac{1}{6}a$, $y=\frac{1}{3}a$, and $z=\frac{1}{2}a$. Ans.

Corollary. *Into whatever number of parts* a *be divided, the divisions will be in the direct ratio of the powers.*

Ex. 5. Put x for the value of the fraction sought, then by the question x^m-x^n is a maximum, and for that reason $mx^{m-1}=nx^{n-1}$. Now because m must be to n in the ratio of 1 to *some* number whether integral or fractional, if m be assumed as unity, whilst n represents the number to which 1 has the same ratio as m to n, it will be $1=nx^{n-1}$, and by transposition, &c. $x=\sqrt[n-1]{\frac{1}{n}}$; wherefore $\sqrt[n-1]{\frac{m}{mn}}$ *is the fraction required.*

Ex. 6. Because the fluxions of the two parts must be at all times equal, the one negative and the other positive, if x and y be assumed, it will be $\dot{x}=-\dot{y}$, and $-\dot{x}=\dot{y}$. But by the question $2x^2+xy+3y^2$ is a minimum, therefore (taking the fluxion and rejecting the fluxional letters,) $4x+y-x-6y$, that is, $3x-5y$, is a minimum. Now $x+y$ being equal to 80, there may be substituted $80-x$ for y, or $80-y$ for x. Consequently $3x=400-5x$; or $240-3y=5y$. Hence $x=50$, and $y=30$. Ans.

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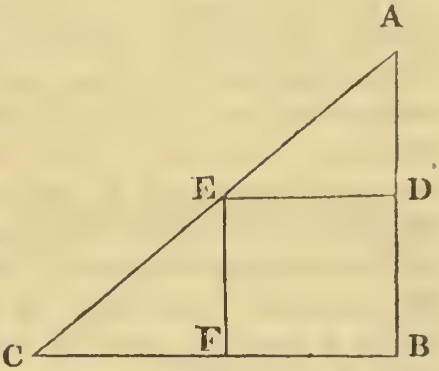
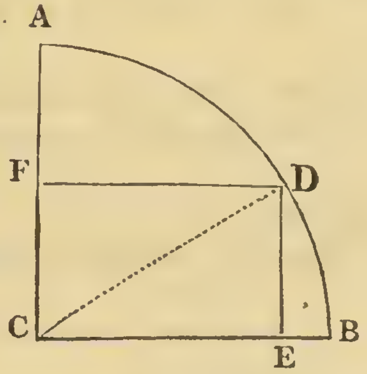
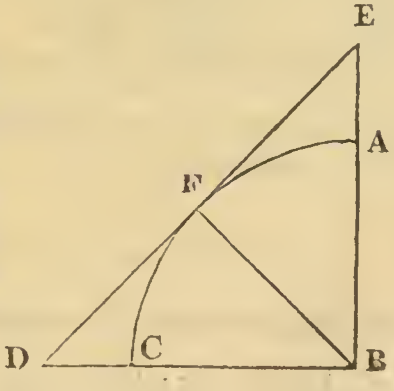
* Signifying that either part must be to the other as the power of the former to the power of the latter. Whereby it is evident that $x=am\div(m+n)$ and $y=an\div m+n)$.



Ex. 7. Let ABC be the triangle, and draw DE parallel to BC, and EF parallel to AB. Also put a for AB the perpendicular, b for BC the base, and x for BD the altitude of the rectangle sought. Then AD $=a-x$. But, As $a : b :: (a-x) : (\frac{ba-bx}{a})=$ DE; therefore $\frac{bax-bx^2}{a}=\frac{b}{a}\times(ax-x^2)=$ the area of the rectangle required. Now the fluxion of $ax-x^2=a\dot{x}-2x\dot{x}=0$, by the question; hence $x=\frac{1}{2}a$, or the rectangle will have half the altitude of the triangle. Q. E. I.



Ex. 8. In the quadrant ABC, draw FD parallel to CB; DE parallel to AC, and join CD. Also put r for the radius CD, and x for CE. ED $=\sqrt{r^2-x^2}$, and CE $\times$ ED $=$ area of the rectangle $=x\sqrt{r^2-x^2}$; of which the fluxion is $\frac{r^2\dot{x}-2x^2\dot{x}}{\sqrt{r^2-x^2}}$. Whence $r^2=2x^2$, or $x=r\sqrt{\frac{1}{2}}=r\times{}$ ·70710678 &c. Ans.



Ex. 9. Ans. When the sides about the right angle are equal, and the hypothenuse a tangent to the circle at the middle point of the arc. See the annexed diagram, where ABC represents the quadrant, F the middle point of the arc, DE a tangent to the circle at F, and DBE the triangle required.

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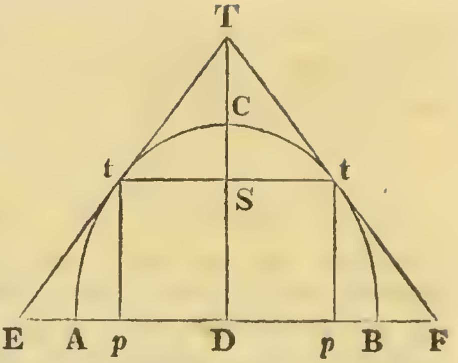

Ex. 10. Ans. *For the triangle.* When the isoceles sides are tangents to the curve, and the altitude of the triangle double the length of the subtangent.

For the rectangle. When the altitude is half the altitude of the circumscribing triangle; which is evident by Ex. 7, in maxima and minima. See the figure in the margin, where ACB is the semi-ellipse; CD half the transverse diameter; ET and FT tangents to the curve at the points *t*, *t*; ST the subtangent equal to half of TD; AB the conjugate diameter produced both ways to meet the tangents TF and TE in the points F and E; ETF the required triangle; and *tppt* the inscribed rectangle.

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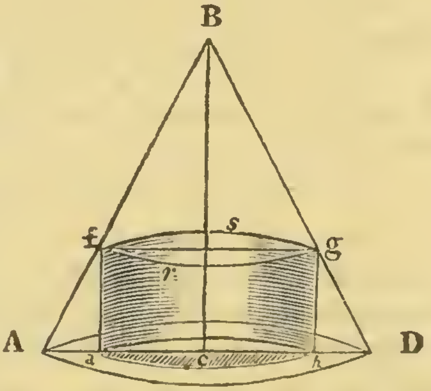
Ex. 11. Ans. The triangle and rectangle will be precisely as in the semi-ellipse in Ex. 10.

Ex. 12. Ans. The same limitations as given in the Answer to the last two Questions.



Ex. 13.

Put a for BC, the altitude of the cone;

b for AD, the diameter of the base;

x for $fg = dh$, the diameter of the cylinder;

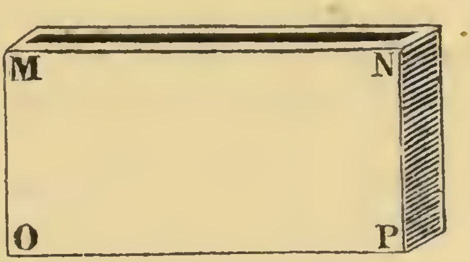
and m for $\frac{3\cdot1416}{4} = \cdot7854$ the area of a circle whose diameter is 1.

Then,

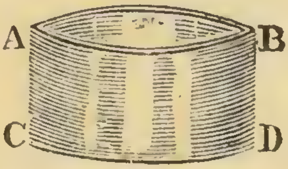
As $1^2 : x^2 :: m : mx^2 =$ the area of the circle $fsgr$.

Again, because the triangle Adf is similar to the triangle ACB,

It is, As $\frac{1}{2}b : a :: (\frac{1}{2}b - \frac{1}{2}x) : df = \frac{ab - ax}{b}$; which multiplied by the area of the cylindrical base, $mx^2 = \frac{ambx^2 - amx^3}{b} =$ the solidity of the cylinder and a maximum. Therefore $\frac{2abmx\dot{x} - 3amx^2\dot{x}}{b} = 0$, or $x = \frac{2}{3}b$; and $df = \frac{1}{3}a$. Hence the diameter of the greatest cylinder that can be inscribed in any cone is $\frac{2}{3}$ of the diameter of the base of the cone, and the altitude of the cylinder $\frac{1}{3}$ the altitude of the cone. *Which was required.*



Ex. 14. By Example 1. in maxima it is evident that the base of the cistern must be a square, therefore put x for the side of the base, OP; and y for the altitude of the cistern, OM. Then $4x=$the periphery of the base, $x^2=$area of the base, and $4xy=$internal superficies of the sides of the cistern. Also the solidity is x^2y, which put$=a$ the quantity of water given. Now since $x^2y=a$, and $4xy$ the internal superficies of the sides, it follows that $\frac{4}{x} \times \frac{x^2y}{1} = \frac{a}{1} = 4xy = \frac{4a}{x} =$ the internal superficies of the sides. Therefore $x^2 + \frac{4a}{x} =$ the whole part to be covered with lead and a minimum. But the fluxion of $x^2 + 4ax^{-1}$ is $2x\dot{x} - 4ax^{-2}\dot{x} = 0$. Hence $2x^3 = 4a$, or $x^3 = 2a$, and $x = \sqrt[3]{2a}$. Again, since $x^2y = a$, and $x^3 = 2a$, it follows that, $2x^2y = x^3$, or $2y = x$; consequently the internal side of the base must be double the altitude. Q. E. I.



Ex. 15. For one quart ale measure in cubic inches put a, and let x represent the diameter of the tankard, CD, or AB; y the depth, AC, or BD. Moreover put $m=3{\cdot}1416$ the circumference of a circle of which the diameter is unity. Then, As $1 : m :: x : xm$, the circumference of the base: consequently xym is the concave superficies of the cylinder. Again the area of the base $= \frac{mx^2}{4}$ which drawn into the altitude y, gives $\frac{mx^2y}{4} =$ the solidity $= a$. Hence, (as in the last example.) $\frac{4a}{x} + \frac{mx^2}{4} =$ the quantity of silver a minimum.

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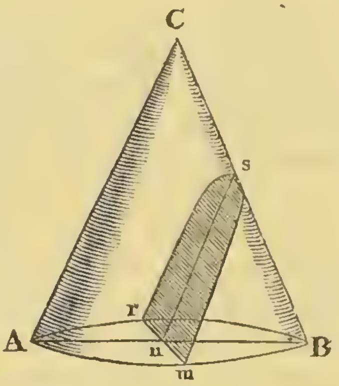
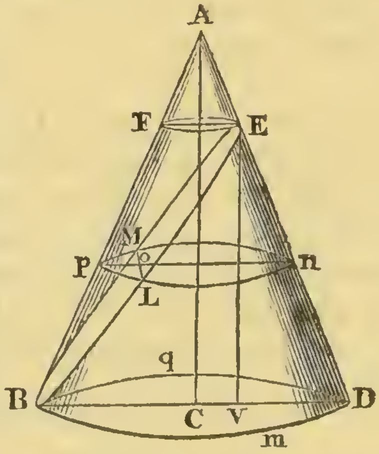
But the fluxion of $4ax^{-1}+\frac{1}{4}mx^2$ is $\frac{mx\dot{x}}{2}-\frac{4a\dot{x}}{x^2}=0$. Therefore $mx^3=8a$, and $x=2\sqrt[3]{\frac{a}{m}}$. Now $\frac{mx^2y}{4}$ being equal to a, and mx^3 equal to $8a$, it follows that, $2mx^2y=mx^3$; or, dividing by mx^2, $2y=x$; that is, half the diameter of the base is equal the altitude. But $a=70\frac{1}{2}$ cubic inches; wherefore $y=2\cdot8205$ inches, and $x=5\cdot641$ inches. Q. E. I.



Ex. 16. Let the axis of the parabola be ns parallel to AC, and let rm be the ordinate or base. Also put a for AB; b for AC; and x for Bn. Then, As $a : b :: x : a^{-1}bx=ns$; likewise $rn\times rn=\mathrm{B}n\times n\mathrm{A}=\overline{nm}|^2=\overline{a-x}\times x=ax-x^2$. Therefore $rm=2\sqrt{ax-x^2}$. But every parabola is $\frac{2}{3}$ of a parallelogram of the same base and altitude, consequently $\frac{2}{3}\times rm\times ns$=area of the parabola $=\frac{2}{3}\times\frac{bx}{a}\times2\sqrt{ax-x^2}=\frac{4bx}{3a}\sqrt{ax-x^2}$; of which the fluxion is $3ax^2\dot{x}-4x^3\dot{x}=0$. Hence $x=\frac{3}{4}a$, $rm=\mathrm{AB}\times\sqrt{\frac{3}{4}}$, and $ns=\frac{3}{4}$ of AC. Ans.



Ex. 17. Let ML represent the less, and BE the greater axis of the ellipse; and let Ev be parallel to AC the axis of the cone, meeting BD the diameter of the base in v. Also let the diameters EF and pn be parallel to BD, and such that pn may pass through o the center of the ellipse considered as variable by the motion of E on the line AD. If a be now put for the axis of the cone, b for the radius of the cone's base, and x for Cv, then will $b+x=\mathrm{BC}+\mathrm{C}v=\mathrm{B}v$ And, As $b : a :: b-x : \frac{ab-ax}{b}=\mathrm{E}v$. Wherefore (Euc. B. I. Prop. xlvii.) $\mathrm{BE}=\sqrt{a^2\times\overline{b-x}|^2+b^2\times\overline{b+x}|^2}$. Moreover because of the similar triangles BEF and BOp, EBD and EOn, and that $\mathrm{OB}=\mathrm{OE}=\frac{1}{2}\mathrm{BE}$, O$n=b$; likewise $op=x$. But

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$op \times on = \text{OL}^2 = bx$; that is $\text{ML} = 2\sqrt{bx}$; therefore $\text{BE} \times \text{ML} = \dfrac{\sqrt{4bx \times \overline{a^2 \times \overline{b-x}|^2 + b^2 \times \overline{b+x}|^2}}}{b}$.

Now the area of an ellipse being in the constant ratio of the rectangle of its two diameters [*Cor.* 2. *Theor.* iii. *of the Ellipse, Conic Sections.*], that is, in the ratio of ·78539816339744830961566084581987572104929234984377645524373614807695410 1 &c. to 1; and the area being at the last given expression a *maximum*, the expression itself is a *maximum*: for which reason its fluxion $b^4\dot{x} + 4b^3x\dot{x} + 3b^2x^2\dot{x} + a^2b^2\dot{x} - 4a^2bx\dot{x} + 3a^2x^2\dot{x} = 0$. Consequently $x^2 - \dfrac{4bx \times (a^2-b^2)}{3a^2+3b^2} = \dfrac{-b^2}{3}$, or $x = \dfrac{2b \times \overline{a^2-b^2}}{3a^2+3b^2} \pm \dfrac{b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$, by which the ellipse is determined.

If, however, $a^4 - 14a^2b^2 + b^4$ be, as it sometimes may, negative, the solution (since the root of a negative quantity is impossible) in that case fails. To determine the limit, suppose that $a^4 - 14a^2b^2 + b^4 = 0$, or, which is the same, put $a^4 + b^4 = 14a^2b^2$; then $a^2 = b^2 \times \overline{\sqrt{48}+7}$, and $a = b \times \overline{2+\sqrt{3}}$; consequently, As $a : b :: (2+\sqrt{3}) : 1$. Hence if the ratio of the altitude of the cone to half the diameter of the base, be not greater than that of $2+\sqrt{3}$ to 1, that is, if the angle ADB be not greater than 75°, the fluxion of the ellipse cannot be equal to 0, for the area would constantly increase as E approaches to D, and be greatest when E is at D.

Again, if the ellipse do not *increase continually* from the vertex to the base of the cone, as in the case when the angle ADC is not greater than 75°, the elliptical area will increase till it arrives at a certain point determinable, afterwards decrease, and again increase till it coincides with the base of the cone. [*See Simpson's Fluxions, where it is proved that, unless the angle which the slant side makes with the axis of the cone be less than* 11°·57′, *the greatest ellipse will be less than the base of the cone.*]

Ex. 18. The fluxion of the hyp. log. of x^x (*See Art.* 68, *page* 327, *and Ex.* 57, *page* 300, vol. ii.) is $\dot{x} + \dot{x}$ (hyp. log. of x.) which by the question is a maximum, and therefore equal to 0; hence the hyp. log. of $x = -1$. Now the number answering to any hyperbolic logarithm z,* (see page 328, vol. ii.) is $1 + z + \dfrac{z^2}{2} +$

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* Let z be denoted by $\dfrac{1+n}{1}$, that is by $1+n$.

(Key to Vol. II. page 319.)

$\frac{z^3}{2\cdot3}+\frac{z^4}{2\cdot3\cdot4}+$ &c.; consequently writing -1 for z, there arises this equation, $x=1-1+\frac{1}{2}-\frac{1}{2\cdot3}+\frac{1}{2\cdot3\cdot4}-\frac{1}{2\cdot3\cdot4\cdot5}+$ &c. which series summed $=0\cdot367878$ &c. Ans.

THE METHOD OF TANGENTS.

(Page 320.)

Ex. 2. $ax-x^2$ being equal to y^2, their fluxions $a\dot{x}-2x\dot{x}$ and $2y\dot{y}$ are equal, therefore $\frac{\dot{x}}{\dot{y}}=\frac{2y}{a-2x}=\frac{y}{\frac{1}{2}a-x}$, or $\frac{y\dot{x}}{\dot{y}}=\frac{y^2}{\frac{1}{2}a-x}=$ the subtangent. Ans.

Ex. 3. ax being equal to y^2, their fluxions $a\dot{x}$ and $2y\dot{y}$ are equal, hence $\frac{\dot{x}}{\dot{y}}=\frac{2y}{a}$, and $\frac{y\dot{x}}{\dot{y}}=\frac{2y^2}{a}=\frac{2ax}{a}=2x$; that is, the subtangent is double its corresponding absciss. Q. E. I.

Ex. 4. The equation being $c^2(ax-x^2)=a^2y^2$, the fluxion of the one side is equal to the fluxion of the other; wherefore $c^2\times\overline{a\dot{x}-2x\dot{x}}=2a^2y\dot{y}$, or $\frac{\dot{x}}{\dot{y}}=\frac{2a^2y}{c^2(a-2x)}$. Consequently $\frac{y\dot{x}}{\dot{y}}$ $=\frac{2a^2y^2}{c^2(a-2x)}=\frac{a^2y^2}{c^2(\frac{1}{2}a-x)}=\frac{c^2(ax-x^2)}{c^2(\frac{1}{2}a-x)}=\frac{ax-x^2}{\frac{1}{2}a-x}$, which was required.

Ex. 5. The equation of the curve being $c^2(ax+x^2)=a^2y^2$, it follows that, $c^2(a\dot{x}+2x\dot{x})=2a^2y\dot{y}$; hence $\frac{y\dot{x}}{\dot{y}}=\frac{a^2y^2}{c^2(\frac{1}{2}a+x)}=$ $\frac{ax+x^2}{\frac{1}{2}a+x}=$ the subtangent. Wherefore the point through which the tangent must pass being determined, the tangent may be drawn. Q. E. F.

(Key to Vol. II. page 320.)

Ex. 6. Because $xy=a^2$, it is manifest that $x=\frac{a^2}{y}=a^2y^{-1}$. Now taking the fluxion of this equation, $\dot{x}=\frac{-a^2\dot{y}}{y^2}$ and $\frac{\dot{x}}{\dot{y}}=\frac{-a^2}{y^2}$; consequently $\frac{y\dot{x}}{\dot{y}}=\frac{-a^2}{y}$ = the value of the sub-tangent, which being a negative quantity, indicates that the tangent and vertex will be on different sides of the ordinate, but the sub-tangent equal to the absciss. Q. E. I.

THE LENGTH OF CURVE LINES.

(Page 323.)

Ex. 2. $x=\frac{y^n}{a^{n-1}}$ being a general expression for the parabola, $\dot{x}=\frac{ny^{n-1}\dot{y}}{a^{n-1}}$; and because $\dot{z}=\sqrt{\dot{y}^2+\dot{x}^2}$, it is evident that, $\dot{z}=\sqrt{\dot{y}^2+\frac{n^2y^{2n-2}\dot{y}^2}{a^{2n-2}}}=\dot{y}\sqrt{1+\frac{n^2y^{2n-2}}{a^{2n-2}}}$, of which the fluent in an infinite series is $y+\frac{n^2y^{2n-1}}{\overline{2n-1}\times 2a^{2n-2}}-\frac{n^4y^{4n-3}}{\overline{4n-3}\times 8a^{4n-4}}+\frac{n^6y^{6n-5}}{\overline{6n-5}\times 16a^{6n-6}}-$ &c. $=z$. Ans.

Ex. 3. Because $ax^2=y^3$, it follows that $x=\frac{\sqrt{y^3}}{\sqrt{a}}$, and $\dot{x}=\frac{3y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}}$. Wherefore $\dot{z}$ (=$\sqrt{\dot{y}^2+\dot{x}^2}$ *as in the last example*) = $\sqrt{\dot{y}^2+\frac{9y\dot{y}^2}{4a}}=\frac{\dot{y}\sqrt{4a+9y}}{2\sqrt{a}}$. Now the fluent of this fluxion is $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$ which corrected by taking $y=0$, becomes $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}-\frac{8a}{27}=z$. Ans.

(Key to Vol. II. page 323.)

Ex. 4. Let the semi-transverse axis of the ellipse be represented by *a*, and the semi-conjugate by *b*, then the equation of the curve is $\frac{a^2y^2}{b^2} = 2ax - x^2$. Hence $x = a \pm \sqrt{a^2 - \frac{a^2y^2}{b^2}} = a \pm \frac{a\sqrt{b^2 - y^2}}{b}$, and $\dot{x} = \frac{-ay\dot{y}}{b\sqrt{b^2-y^2}}$. Again, $\dot{z}$ $(=\sqrt{\dot{x}^2 + \dot{y}^2}.) = \sqrt{\dot{y}^2 - \frac{a^2y^2\dot{y}^2}{b^2(b^2-y^2)}} = \dot{y}\sqrt{1 - \frac{a^2y^2}{b^4 - b^2y^2}}$. Now converting $\frac{a^2y^2}{b^4 - b^2y^2}$ to an *infinite series*, the last fluxion becomes $\dot{y} \times \sqrt{1 - \frac{a^2y^2}{b^4} + \frac{a^2y^4}{b^6} - \frac{a^2y^6}{b^8} + \&c.} = \dot{y} - \frac{a^2}{2b^4}y^2\dot{y} + (\frac{a^2}{2b^6} + \frac{a^4}{8b^8})\,y^4\dot{y} - (\frac{a^2}{2b^8} + \frac{a^4}{4b^{10}} + \frac{a^6}{16b^{12}})\,y^6\dot{y} + \&c.$ Wherefore $z = y - \frac{a^2y^3}{6b^4} + (\frac{a^2}{b^2} + \frac{a^4}{4b^4}) \times \frac{y^5}{10b^4} - (\frac{a^2}{b^2} + \frac{a^4}{2b^4} + \frac{a^6}{8b^6}) \times \frac{y^7}{14b^6} + \&c.$ Ans.

Ex. 5. If *a* and *b* respectively be assumed, as in the last Example, for the semi-transverse and semi-conjugate axes of the hyperbola, the equation of the curve will be $\frac{a^2y^2}{b^2} = 2ax + x^2$; hence $x = \frac{a\sqrt{b^2+y^2}}{b} - a$, and $\dot{x} = \frac{ay\dot{y}}{b\sqrt{b^2+y^2}}$. Proceeding, therefore, in all respects as in the last Example, $z = y + \frac{a^2y^3}{6b^4} - \frac{y^5}{10b^4}(\frac{a^2}{b^2} + \frac{a^4}{4b^4}) + \frac{y^7}{14b^6}(\frac{a^2}{b^2} + \frac{a^4}{2b^4} + \frac{a^6}{8b^6}) - \&c.$ Ans.

QUADRATURES.

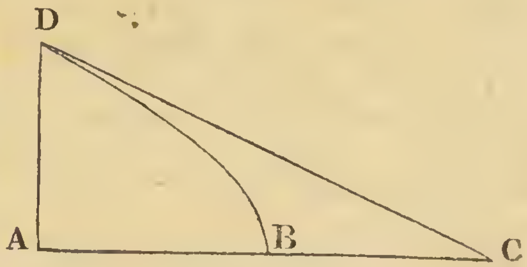
(Page 325.)

Ex. 3. Extracting the $(m+n)th$ root on both sides, the general equation becomes $y = a^{\frac{m}{m+n}} z^{\frac{n}{m+n}}$; and putting $\dot{w}$ for $y\dot{z}$ there arises $\dot{w} = a^{\frac{m}{m+n}} z^{\frac{n}{m+n}} \dot{z}$. Wherefore w (the fluent or area) is $a^{\frac{m}{m+n}} \times \frac{z^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} = \frac{a^{\frac{m}{m+n}} z^{\frac{n}{m+n}} (m+n)z}{m+2n} = \frac{m+n}{m+2n}$ times the rectangle under the absciss and ordinate. Ans.

Cor. 1. When $m=n$, the figure is the common parabola, and the area $=\frac{2}{3}zy$. $(=\frac{2}{3}xy.)$

Cor. 2. When $m=0$, the figure is a right angled isoceles triangle, and its area $=\frac{1}{2}zy$. $(=\frac{1}{2}xy.)$

Ex. 4. Let a represent the transverse axis, and c the conjugate, so shall $y=\frac{c}{a}\sqrt{ax-x^2}$, and $\dot{w} = \frac{c}{a} \times \dot{x}\,(ax-x^2)^{\frac{1}{2}}$. Now the fluxion of w is the fluxion of the area bounded by the curve, absciss, and ordinate, to any portion of the elliptic curve having the transverse diameter a for its axis, x for its absciss, and y for its ordinate. But $\dot{x}\ (ax-x^2)^{\frac{1}{2}}$ [*see Ex.* 2. *page* 324, *vol.* ii.] is the fluxion of the corresponding area of a circle of which the diameter is a. Again, the fluent of $\dot{x}\,(ax-x^2)^{\frac{1}{2}}$ is $x\sqrt{ax} \times (\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \frac{3x^3}{216a^3} - \frac{15x^4}{2112a^4} -$ &c.) And the fluent of w is $\frac{c}{a} \times x\sqrt{ax} \times (\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2}$ $-$ &c.) Consequently the area of any segment of an ellipse is, to the area of the corresponding segment of the circumscribing circle, as the less axis of the ellipse is to the diameter of that circle; and the wholes in the same ratio. Q. E. D.



Ex. 5. Let B in the annexed figure be the vertex of an hyperbola, CD a tangent to the curve at the point D, BC the semi-transverse axis, and AB one of the abscisses to the ordinate AD. Also put a for BC, b for the semi-conjugate axis of the hyperbola, x for the subtangent AC, and y for the ordinate AD. Then by the property of the curve, $y = \frac{b}{a}\sqrt{x^2-a^2}$.

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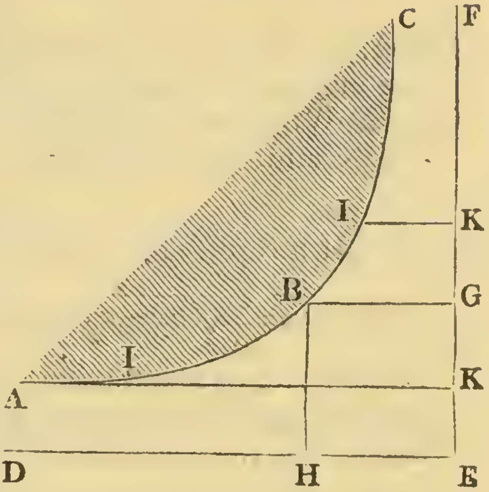
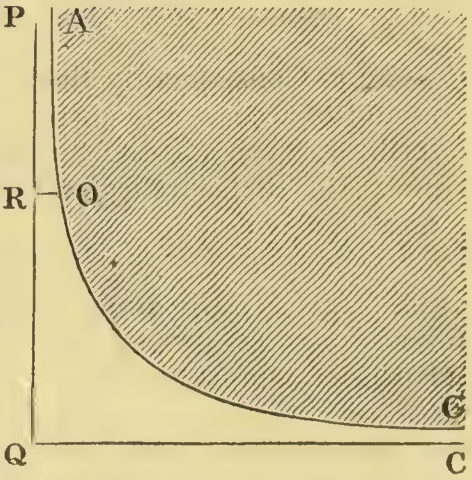
Wherefore $\dot{w}$, that is, $y\dot{x} = \frac{c\dot{x}}{a}\sqrt{x^2-a^2}$ the fluxion of the hyperbolic area, of which the correct fluent or area is $\frac{cx\sqrt{x^2-a^2}}{2a} - \frac{1}{2}ac \times$ hyp. log. $\frac{x+\sqrt{x^2-a^2}}{a}$. Q. E. I.



Ex. 6. Let AIBIC be an hyperbola, DE and EF the asymptotes, B the vertex, and GBIK the required space between the curve and asymptote. Also put a for EK, x for GK, and y for IK. Then by the property of the curve, $y = \frac{a^2}{a+x}$.

Now if a be considered unity, the value of y is $\frac{1}{1+x}$, and $y\dot{x} = \frac{\dot{x}}{1+x}$ = the fluxion of the area sought. But $\frac{\dot{x}}{1+x} = \dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x} -$ &c. in an *infinite series*, and the fluent of this fluxion is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 -$ &c. the area GBIK. Q. E. I.



Ex. 7. Let the curve be AOC, the asymptotes PQ, QC, and the area of quadrature CCQRO. Then by the given equation, $y = \frac{a^{\frac{m+n}{n}}}{\sqrt[n]{x^m}} = a^{\frac{m+n}{n}} x^{\frac{-m}{n}}$, consequently $y\dot{x} = a^{\frac{m+n}{n}} x^{\frac{-m}{n}} \dot{x}$. Now of this fluxion the fluent is $\frac{a^{\frac{m+n}{n}} x^{\frac{n-m}{n}}}{\frac{1}{n}(n-m)} = \frac{na^{\frac{m+n}{n}} x^{\frac{n-m}{n}}}{n-m}$ the correct area when x is 0 and n greater than m. But if n be less than m when x is 0, the area is infinite, because the index of x being negative, x (that is, 0) descends to the denominator, and $na^{\frac{m+n}{n}}$ a finite quantity is divided by 0.

In this case, however, the area APRO will be finite, and its

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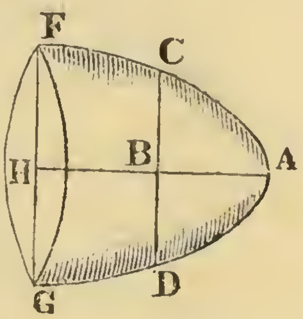
fluxion $= -a^{\frac{m+n}{n}} x^{\frac{-m}{n}} \dot{x}$; of which the fluent (reduced, and its signs changed) is $\frac{na^{\frac{m+n}{n}} x^{\frac{n-m}{n}}}{m-n}$, evidently wanting no correction because vanishing when x is infinite. Q. E. I.

☞ *This example may be seen at some length, and treated in a different manner, in* VINCE's *excellent treatise on Fluxions,* 4th edit. p. 70.

THE SURFACES OF SOLIDS.

(Page 326.)

Ex. 2. 1. *For an Oblong Spheroid.*



Let FCADG be a semi-spheroid generated by the rotation of the quarter ellipse HFCA about the semi-transverse axis HA; and let CBD be a plane parallel to the base of the semi-spheroid meeting the transverse axis in any point B. If a be put for AH, b for FH, x for BH, y for BC, z for FC, and s for the superficies generated by FC, it will be, by the property of the ellipse, $y = \frac{b}{a}\sqrt{a^2 - x^2}$. Hence $\dot{y} = \frac{-bx\dot{x}}{a\sqrt{a^2 - x^2}}$, and $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{x}^2 + \frac{b^2x^2\dot{x}^2}{a^2(a^2 - x^2)}} = \frac{\dot{x}\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}}$. Now substituting e for $(a^2 - b^2)^{\frac{1}{2}}$ the eccentricity of the ellipse, the last fluxion becomes $\frac{\dot{x}\sqrt{a^4 - e^2x^2}}{a\sqrt{a^2 - x^2}} = \frac{e\dot{x}\sqrt{a^4e^{-2} - x^2}}{a\sqrt{a^2 - x^2}}$. Consequently $\dot{s}$, that is, $2\overset{*}{c}y\dot{z}$ or $2cy\sqrt{\dot{x}^2 + \dot{y}^2}, = \frac{2bce\dot{x}\sqrt{a^4e^{-2} - x^2}}{a^2}$, of which the fluent† expressed in an infinite series is $2bcx\,(1 - \frac{e^2x^2}{6a^4} - \frac{e^4x^4}{40a^8} - \frac{e^6x^6}{112a^{12}} -$ &c. *in infinitum.*)

* *Where* c *represents the circumference of the circle of which the diameter is* UNITY.

† Which might also be expressed in finite terms.

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2. *For an Oblate Spheroid.*

Because AH *(see the last figure)* is in an oblate spheroid less than FH, $\sqrt{a^2-b^2}$ is an imaginary value, the square root of a negative quantity being impossible. In this case, therefore, put $e=\sqrt{b^2-a^2}$, and take $m : a :: a : e$, so shall $m=a^2e^{-1}$. Hence $\dot{s}=\frac{2bce\dot{x}\sqrt{a^4e^{-2}+x^4}}{a^2}=\frac{2bc\dot{x}\sqrt{m^2+x^2}}{m}=\frac{2bc}{m}\times\dot{x}\sqrt{m^2+x^2}$, of which the fluent is $\frac{bcx}{m}\sqrt{m^2+x^2}+bcm\times$ hyp. log. $\frac{x+\sqrt{m^2+x^2}}{m}$.

Ex. 3. If a denote the parameter of the axis, the equation of the generating curve will be $ax=y^2$; hence $x=\frac{y^2}{a}$, and $\dot{x}=\frac{2y\dot{y}}{a}$. Consequently $\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{4a^{-2}y^2\dot{y}^2+\dot{y}^2}=\frac{\dot{y}\sqrt{4y^2+a^2}}{a}$, and $2cy\dot{z}=\frac{2cy\dot{y}}{a}\times(4y^2+a^2)^{\frac{1}{2}}$, of which the fluent is $\frac{c\overline{\sqrt{4y^2+a^2}}|^{\frac{3}{2}}}{6a}$ or, (when corrected by taking $y=0$,) $\frac{c\overline{\sqrt{4y^2+a^2}}|^{\frac{3}{2}}}{6a}-\frac{1}{6}a^2c$. Ans.

Ex. 4. Put a for the semi-transverse axis of the generating hyperbola, b for the semi-conjugate, and x for the distance from the center of any ordinate to the curve. Then, by the property of the hyperbola, $y=\frac{b}{a}\sqrt{x^2-a^2}$; hence $\dot{y}=\frac{bx\dot{x}}{a\sqrt{x^2-a^2}}$, and $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{x}\sqrt{a^2+b^2}\times(x^2-a^4)}{a\sqrt{x^2-a^2}}$. Consequently $2cy\dot{z}=\frac{2bc\dot{x}\sqrt{a^2+b^2}\times(x^2-a^4)}{a^2}$; of which the fluent (assuming $n^2=\frac{a^4}{a^2+b^2}$) is $\frac{bcx\sqrt{x^2-n^2}}{n}-bcn\times$ hyp. log. of $\overline{x+\sqrt{x^2-n^2}}$. Lastly, correcting this fluent by

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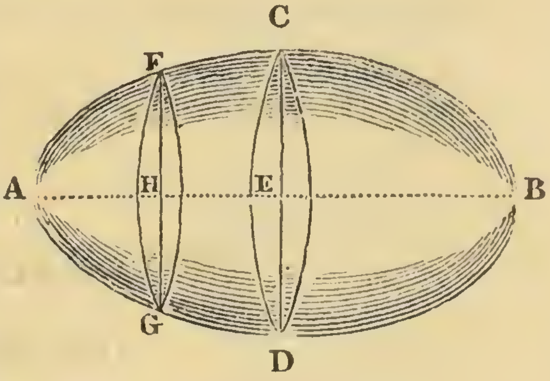
putting $x=a$, the true fluent is $(\frac{bcx}{n}\sqrt{x^2-n^2}-b^2c)-bcn$

$\times$ hyp. log. of $\frac{x+\sqrt{x^2-n^2}}{(a^2+bn)\,a^{-1}}=(\frac{bcx}{n}\sqrt{x^2-\frac{n^2}{2x}}-b^2c)-bcn$

$\times$ hyp. log. of $\frac{a}{a^2+bn}(2x-\frac{n^2}{2x}-\frac{n^4}{8x^3}-\frac{n^6}{16x^5}-\frac{5n^8}{128x^7}-\&c.)$

THE SOLIDITY OF BODIES.

(Page 327.)



Ex. 2. Let AB be the transverse and CD the conjugate axis of the generating ellipse; also let FHG be a plane at right angles to, and cutting the transverse axis in any point H. Then if the greater axis of the solid be represented by a, the less by b, AH by x, and HG by y, it will be, by the property of the ellipse, As $a^2 : b^2 :: x(a-x) : y^2$, whence $y^2=\frac{b^2}{a^2}(ax-x^2)$, and $cy^2\dot{x}=a^{-2}b^2c\,(ax\dot{x}+x^2\dot{x})$, of which the fluent is $a^{-2}b^2c\,(\frac{1}{2}ax^2-\frac{1}{3}x^3)$ the content of the segment BGF. Now when H coincides with B, x is equal to a, and $a^{-2}b^2c\,(\frac{1}{2}ax^2-\frac{1}{3}x^3)$ is the fluent of the whole spheroid, which, by substituting a for x, becomes $a^{-2}b^2c\,(\frac{1}{2}a^3-\frac{1}{3}a^3)=\frac{1}{2}ab^2c-\frac{1}{3}ab^2c=\frac{1}{6}ab^2c$.

Corollary 1. When AB=CD, the solid is a sphere, and its content truly expressed by $\frac{1}{6}a^3c$.

Corollary 2. A sphere and spheroid, whether oblong or oblate, are each $\frac{2}{3}$ of the solidity of the circumscribing cylinder, for $\frac{1}{4}ab^2c$ being the content of a cylinder of which the diameter of the base is b, and altitude a, it is evident that, As $\frac{1}{6}:\frac{1}{4}::\frac{2}{3}:1$ :: content of a sphere or spheroid : the content of the circumscribing cylinder.

Ex. 3. The equation of the generating curve being $a^{m-n}x^n=y^m$, it follows that $y=a^{\frac{m-n}{m}}x^{\frac{n}{m}}$. Hence $cy^2\dot{x}=ca^{\frac{2m-2n}{m}}x^{\frac{2n}{m}}\dot{x}$, of

(Key to Vol. II. page 327.)

which the fluent is $ca^{\frac{2m-2n}{m}} \times \frac{x^{\frac{2n}{m}+1}}{\frac{1}{m}(2n+m)} = ca^{\frac{2m-2n}{m}} \times \frac{mx^{\frac{2n+m}{m}}}{2n+m} = ca^{\frac{2m-2n}{m}} \times \frac{mx}{2n+m} \times x^{\frac{2n}{m}} = cy^2 \times \frac{mx}{2n+m}$. Ans.

Cor. 1. If $m=2$, and $n=1$, the solid is the common paraboloid, and equal to $\frac{1}{2}$ its circumscribing cylinder.

Cor. 2. If $m=n$, the curve is changed to a straight line, and the solid generated is a cone, equal to $\frac{1}{3}$ its circumscribing cylinder.

Ex. 4. Because the equation of the curve is $\frac{b^2}{a^2}(ax+x^2)=y^2$, it follows that, $cy^2\dot{x} = \frac{b^2c}{a^2}(ax\dot{x}+x^2\dot{x})$ of which the fluent is $\frac{b^2c}{a^2}(\frac{1}{2}ax^2+\frac{1}{3}x^3)$. *The content required.*

OF LOGARITHMS.

(Page 329.)

Ex. 2. The fluxion of $\frac{a+x}{b}$ is $\frac{\dot{x}}{b}$, which divided by $\frac{a+x}{b}$ becomes $\frac{\dot{x}}{a+x} = \dot{x} \times \frac{1}{a+x} = \dot{x} \times (\frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \frac{x^4}{a^5} - \frac{x^5}{a^6} + \&c.)$ $= \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} + \frac{x^4\dot{x}}{a^5} - \&c.$ of which the fluent is $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + \&c.$ *the logarithm reqd.*

Ex. 3. The fluxion of $a-x$ is $-\dot{x}$, which divided by $a-x$ becomes $\frac{-\dot{x}}{a-x} = \dot{x} \times \frac{-1}{a-x} = \dot{x} \times (-\frac{1}{a} - \frac{x}{a^2} - \frac{x^2}{a^3} - \frac{x^3}{a^4} - \frac{x^4}{a^5} - \&c.)$ $= -\frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} - \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} - \frac{x^4\dot{x}}{a^5} - \&c.$ of which the fluent is $-\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4} - \frac{x^5}{5a^5} - \&c.$ IN INFINITUM, *the logarithm required.*

(Key to Vol. II. page 329.)

Ex. 4. Taking $\frac{a+x}{a-x}=3$, the value of a is $2x$; wherefore $\frac{x}{a}=\frac{1}{2}$, and $\frac{x^2}{a^2}=\frac{1}{4}$ a constant factor in every succeeding term of the series. Likewise $2m=2$ for hyperbolic logarithms, and ·8685889638065036553022578 &c. for common logarithms.

Dividing $2m$, therefore, by 2, the quotients successively by 4, and afterwards collecting the whole of the 1*st* quotient, $\frac{1}{3}$ of the 2*nd* quotient, $\frac{1}{5}$ of the 3*rd* quotient, $\frac{1}{7}$ of the 4*th* quotient, $\frac{1}{9}$ of the 5*th* quotient, $\frac{1}{11}$ of the 6*th* quotient, $\frac{1}{13}$ of the &c. there arise

In the one instance.	In the other instance.
1·	·434294481903
·083333333333	·036191206825
·0125	·005428681023
·002232142857	·000969407325
·000434027777	·000188495868
·000088778409	·000038555973
·000018780048	·000008156071
·000004069010	·000001767148
·000000897582	·000000389812
·000000200773	·000000087194
·000000045413	·000000019722
·000000010366	·000000004501
·000000002384	·000000001035
·000000000551	·000000000239
·000000000128	·000000000055*
1·098612288631 = the hyp. log. of 3, true to 10 decimal places.	·477121254700 = the com. log. of 3, true to 10 decimal places.

* Here 6 are carried from the preceding column (which is omitted for the sake of uniformity,) making the amount of this column 70.

Ex. 5. Ans. { The com. log. of 5 is 0·69897000433601880478 6 &c.
The hyp. log. of 5 is 1·6094379127 &c.

Ex. 6. Ans. { The com. log. of 11 is 1·041392685158225040 7 &c.
The hyp. log. of 11 is 2·397895273016 &c.

(Key to Vol. II. page 331.)

POINTS OF INFLEXION.

Ex. 2. It is imagined the given equation was misprinted, $ay = a\sqrt{ax} + xx$ being intended; in which case $x = \frac{1}{4}a$, and $y = \frac{9a}{16}$. Ans.

Ex. 3. Ans. $\begin{cases} x = a\left(\sqrt{\tfrac{4}{3}} - 1\right)^{\frac{1}{2}} \\ y = a\sqrt{\sqrt{\sqrt{\tfrac{4}{3}} - 1} + \left(\sqrt{\tfrac{4}{3}} - 1\right)^{\frac{3}{2}}} \end{cases}$

Ex. 4. Put a for PF, b for EI=AF, x for FD, and y for DE. Then $x^2 = (a+y)^2 \times (b^2 - y^2) \times y^{-2}$. Whence $y^3 + 3ay^2 = 2ab^2$, an equation in which y is determinable. Now if q be assumed as the value of y in this cubic equation, the value of x is $(a^2b^2q^{-2} + 2ab^2q^{\frac{1}{2}} + b^2 - a^2 - 2aq - q^2)^{\frac{1}{2}}$; and if $a = b$, then $y = a\sqrt{3} - a$, and $x = (a^4q^{-2} + 2a^3q^{\frac{1}{2}} - 2aq - q^2)^{\frac{1}{2}}$. Ans.

RADIUS OF CURVATURE.

(Page 333.)

Ex. 2. Here $y = \frac{c}{a}\sqrt{ax - x^2}$; hence $\dot{y} = \frac{c^2\dot{x}(a-2x)}{2a^2y} = \frac{(ac+2cx)\dot{x}}{2a\sqrt{ax - x^2}}$, and $-\ddot{y} = \frac{a^2\dot{y}^2 - c^2\dot{x}^2}{a^2y} = \frac{ac\dot{x}^2}{4(ax-x^2)^{\frac{3}{2}}}$.

Wherefore $\sqrt{\dot{y}^2 + \dot{x}^2} = \sqrt{\frac{c^2\dot{x}^2(a-2x)^2}{4a^2(ax-x^2)} + \dot{x}^2} = \frac{\dot{x}}{2a} \times \frac{\sqrt{a^2c^2 + 4(ax-x^2)\times(a^2-c^2)}}{ax - x^2} = \dot{z}$; whence it follows that

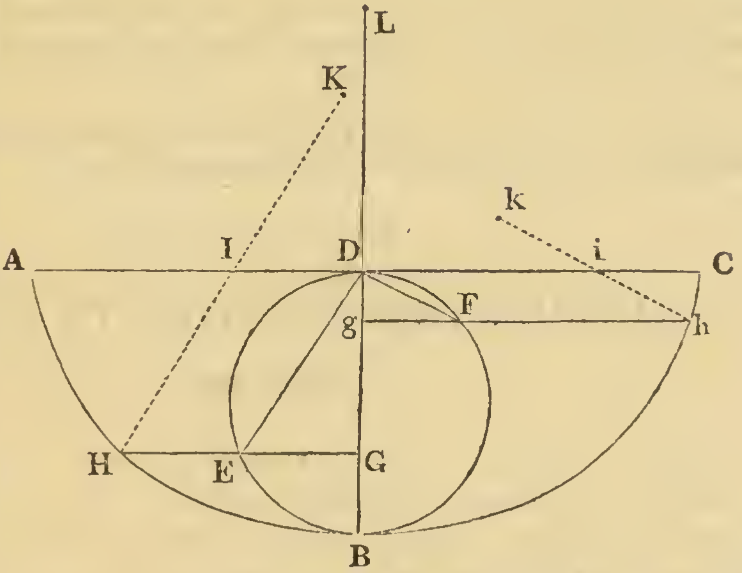
$$\frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\overline{a^2c^2 + 4(ax-x^2)\times(a^2-c^2)}|^{\frac{3}{2}}}{2a^4c}$$ = *the radius of curvature required.*

(Key to Vol. II. page 333.)

Ex. 3. In this Example $y = \frac{c}{a}\sqrt{ax+x^2}$; and $\dot{y} = \frac{(a+2x)c^2\dot{x}}{2a^2y}$ $= \frac{(ac+2cx)x\dot{x}}{2a\sqrt{ax+x^2}}$, also $-\ddot{y} = \frac{ac\dot{x}^2}{4\,(ax+x^2)^{\frac{3}{2}}}$.

Consequently $\sqrt{\dot{y}^2+\dot{x}^2} = \sqrt{\frac{(a+2x)^2c^2\dot{x}^2}{4a^2(ax+x^2)} + \dot{x}^2} =$ $\frac{\dot{x}\sqrt{c^2(a+2x)+4a^2(ax+x^2)}}{4a^2(ax+x^2)} = \frac{\dot{x}}{2a}\sqrt{\frac{a^2c^2+(a^2+c^2)\times 4(ax+x^2)}{ax+x^2}}$ $= \dot{z}$. Wherefore $\frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\overline{a^2c^2+(a^2+c^2)\times 4(ax+x^2)}|^{\frac{3}{2}}}{2a^4c}$ *the radius of curvature required.*

Ex. 4.



Let ABC be a cycloid, BEDF its generating circle, and AC the right line on which the circle rolls. Also let AD be equal to DC, and DB a diameter of the circle, at right angles to AC. Assume in the curve any point H, of which the center and radius of curvature are required; and draw the ordinate HG. If from E the point of intersection of this ordinate and the generating circle, a right line be drawn to D, ED shall equal half the radius of curvature at the point H, and the center K shall be in a straight line parallel to ED, and passing through H; that is, HI shall be half of HK. For, put a for the *radius* of the generating circle, x for the absciss BG, y for the ordinate GH, and w for the arc BE. Then by the property of a circle $GE = \sqrt{2ax-x^2}$; and $y = w + \sqrt{2ax-x^2}$ by the property of a cycloid. Consequently

(Key to Vol. II. page 333.)

$\dot{y}=\dot{w}+\frac{a\dot{x}-xx}{\sqrt{2ax-x^2}}$. Now $\dot{w}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}$, wherefore $\dot{y}=\frac{2a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}=\frac{\dot{x}}{x}\sqrt{2a-x}$; hence $\ddot{y}$, considering $\dot{x}$ constant, is $\frac{-a\dot{x}^2}{x\sqrt{2ax-x^2}}$. These values of $\dot{y}$ and $\ddot{y}$ being substituted in the general expression for $r=\frac{(\dot{x}^2+\dot{y}^2)\times\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}=\frac{(1-\dot{y}^2)\sqrt{1-\dot{y}^2}}{-\ddot{y}}$, the value of r is $2\sqrt{4a^2-2ax}$.* Ans.

Cor. 1. When $x=0$, the radius of curvature is 4 times the radius of the generating circle.

Cor. 2. When $x=2a$, the radius of curvature vanishes.

Hence L *is the center, and* LB *the radius of curvature to the point* B; *and if* h *be assumed as any other point in the curve,* (gh *being the ordinate and* Bg *the absciss,*) hi *is equal and parallel to* FD, *and* kh(=2ih) = *the radius of curvature from the center* k, *to the curve at* h.

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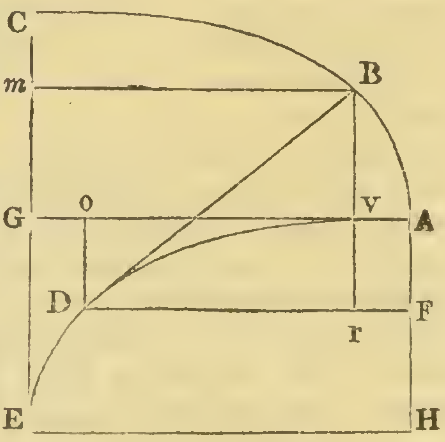
* This expression is equal to $2\sqrt{d^2-dx}$ by considering $a=\frac{1}{2}d$, and recognizing in d the diameter of the generating circle.

INVOLUTE AND EVOLUTE CURVES.

(Page 272.)



Ex. 2. Let ABC be half the curve of a common cycloid, and ADE its evolute, D being the center, and DB the radius of curvature to the point B; also let Bm and FD be parallel to AG the line on which the generating circle of the given curve rolls, whilst oD and Br are at right angles to that line. By the property of the cycloid the arc ABC=2CG. If, therefore, a be put for ABC, x for Av, y for Bv, and z for AB, then BC=$a-z$, C$m=\frac{1}{2}a-y$, and, As $a^2:\frac{1}{2}a::\overline{a-z}^2:\frac{1}{2}a-y$. Consequently

(Key to Vol. II. page 333.)

$y=\frac{2az-z^2}{2a}$, and $\dot{y}=\frac{a\dot{z}-z\dot{z}}{a}$. Now $\dot{y}=\sqrt{\dot{z}^2-\dot{x}^2}$, or $\dot{y}^2=\dot{z}^2-\dot{x}^2$, hence $\dot{x}^2=\frac{\dot{z}^2(2az-z^2)}{a^2}$, $\dot{x}=\frac{\dot{z}\sqrt{2az-z^2}}{a}$, and (considering $\dot{z}$ constant,) $\ddot{x}=\frac{\dot{z}^2(a-z)}{a\sqrt{2az-z^2}}$; hence $\frac{\dot{y}\dot{z}}{\ddot{x}}=\sqrt{2az-z^2}=$ DB $=$ DA.

But when a and z are equal, by the coincidence of EDB with EGC, ADE also coincides with, and is equal to CGE; and GE $=$ AH $=$ GC $=\frac{1}{2}a$. Therefore, because it is plain that, As AE² : AD² : : AH : AF, it follows that, the evolute ADE is a semi-cycloid equal and similar to the involute ABC, but in a contrary direction. *Consequently the evolute of the common Cycloid is a pair of contiguous semi-cycloids in contrary directions, the generating circles of both involute and evolute being equal, and rolling on parallel lines, of which the distance is a diameter of either circle.* Q. E. D.

END OF VOL. II.

KEY

TO

HUTTON's COURSE OF MATHEMATICS.

ISOPERIMETRY.

(Key to Vol. III. page 52.)

Ex. 1. $36 \div 3 = 12$ side of the equilateral triangle.
$36 \div 4 = 9$ side of the square of equal perimeter.
$36 \div 6 = 6$ - - - - - - - - - - hexagon.
$36 \div 12 = 3$ - - - - - - - - - - dodecagon.
Also $\overline{12}^2 \times 0\cdot4330127 = 62\cdot3538288$ area of the eq. lat. triang.
And $9^2 \times 1\cdot0000000 = 81\cdot0000000$ area of the square.
So $6^2 \times 2\cdot5980762 = 93\cdot5307432$ area of the hexagon.
And $3^2 \times 11\cdot1961524 = 100\cdot7653716$ area of the dodecagon.
Lastly $\overline{36}|^2 \times \cdot07958 = 103\cdot1356800$ area of the circle.

Ex. 2. Because a triangle whose sides are 3, 4, 5, is right angled, it follows that,

$3 \times \frac{4}{2} = 3 \times 2 = 6\cdot0000$ is the area of the right angled triangle.

Now $\frac{3+4+5}{3} = 4$ is the side of the equilateral triangle of equal perimeter.

But $4^2 \times 0\cdot4330127 = 6\cdot9282032$ the area of the equilateral triangle. Therefore ·9282032 *is the required difference.*

Ex. 3. By Theor. v. Elem. of Isoperimetry (*see page* 36, *vol.* iii.) the angle contained by the sides 8 and 11 is a right angle; therefore 44 is the area required. And $\sqrt{8^2+11^2} = \sqrt{185} = 13\cdot6014705$ the length of the third side. Q. E. I.

(Key to Vol. III. page 52.)

Ex. 4. $\overline{12}|^2 \times \cdot 07958 = 11\cdot 45952$ the area of the circle.
But (*Theor.* xi. *page* 36. *vol.* iii.) it will be,
As $12 : 15$, ∵ $4 : 5 :: 11\cdot 45952 : 14\cdot 3244$ the area of the polygon.

Whereby the Answer is obvious.

(Page 53.)

Ex. 5. By Theor. xxiii. page 44, vol. iii. the parallelopiped required is a cube; therefore $18 \div 3 = 6 =$ the length of the side of the cube. Consequently $6 \times 6 \times 6 = 216$ is an expression for both the surface and solidity. Q. E. I.

Ex. 6. It is evident by Theor. xxiii. Isoperimetry, and by the last Example that, the square prism proposed is a cube; therefore $546 \div 6 = 91 =$ the surface of any one of the sides. But $\sqrt{91} = 9\cdot 539392 =$ the side of the cube: consequently $\overline{9\cdot 539392}|^3 = 868\cdot 084669421158924288$ *is the solidity required.*

Ex. 7. The surface of the cylinder being a *minimum*, the species (*Theor.* xxvii. *Isoperimetry*) is the Archimedéan. But a cylinder circumscribing a sphere, is to the sphere as $3 : 2$; therefore $113\cdot 097312$ is the solidity of the sphere circumscribed, of which the diameter is 6. Consequently the surface of the cylinder, is expressed by the same figures as its solidity *namely,* by $169\cdot 645968$ &c. Q. E. I.

Ex. 8. Because, by the question, the solidity is a *maximum,* the slant side (*Theor.* xxxiii.) is triple the radius of the base; and consequently the whole surface of the cone quadruple the area of the base. But $201\cdot 061952 \div 4 = 50\cdot 265488$, which divided by $\cdot 78539 = 64$ the square of the diameter of the base of the cone. Hence the radius of the base is 4; the length of the slant side 12; and the altitude $\sqrt{128} = 11\cdot 3137085$, of which $\frac{1}{3}$ part is $3\cdot 77123617$, which $\times^{\text{ing}}$ $50\cdot 265488$ (*the area of the base*)$= 189\cdot 56302643830096$. Ans.

Ex. 9. Since the capacity of the pyramid, given in species, is a *maximum,* it is evident (*Theor.* xxxv.) that, the length of the slant side is triple the radius of a circle inscribed in the base; and (*Theor.* i. *Cor.* 2.) that, the base is equilateral. Now the side of an equilateral triangle circumscribing a circle whose diameter is d, is to that diameter, as $d\sqrt{3}$ to d. Consequently the areas are as $1\cdot 2990381$ to $\cdot 7854$. But (*Theor.* xxxiv.) the ratio of the surfaces of the bases of a right cone and its circumscribing pyramid, is the ratio of the surfaces *total.* Therefore

As 1·2990381 : ·7854 : : 43·30127 : 26·18 the surface of the inscribed cone. Hence, as in the last example, it will be, $26{\cdot}18 \div 4 = 6{\cdot}545$ = the area of the cone's base, which divided by ·78539 = 8·33343 the square of the diameter of the base of the cone. Hence the radius is 1·443 &c. Also the slant side of the cone (and of the triangle) is 4·33. Moreover, the altitude is $\sqrt{\overline{4{\cdot}33}^2 - \overline{1{\cdot}443}^2} = \sqrt{18{\cdot}7489 - 2{\cdot}082249} = \sqrt{16{\cdot}66666} = 4{\cdot}08248$. Again 2·886 (the diameter of the circle inscribed in the base,) × 1·7320508 (the square root of 3) = 4·9986986 = the side of the equilateral triangle forming the base, and therefore the area of the base of the pyramid = 10·782 *nearly.*

Lastly, $\frac{10{\cdot}782 \times 4{\cdot}08248}{3} = 14{\cdot}67243$ &c. *the capacity sought.*

Ex. 10. *For the sphere.* $\overset{(Di.)}{20} \times \overset{(Di.)}{20} \times \overset{(Di.)}{20} \times \overset{(Const.\,dec.)}{{\cdot}5236} = 4188{\cdot}8$ the solidity of the sphere.

For the cone. As 4 : 9 : : 4188·8 : 9424·8 the solidity of the cone.

And *For the cylinder.* As 2 : 3 : : 4188·8 : 6283·2 the solidity of the cylinder. Q. E. I.

Ex. 11. The surface 28·274337 (of the sphere,) divided by 3·141593 = 8·1 = the square of the diameter of the sphere; therefore the diameter is = 2·846. But $28{\cdot}274337 \times 2{\cdot}846 \times \frac{1}{6} = 13{\cdot}410518$ the solidity of the sphere. Now (*Theor.* xxxvi. *Isoperimetry, vol.* iii.) it will be, As 28·274337 : 35 : : 13·410518 : 16·6 the solidity of the irregular polyedron circumscribed about the sphere.

Ex. 12. *For the sphere.* The solidity (500) of the sphere divided by ·5236 gives 954·927 &c. for the cube of the diameter of the sphere, hence the diameter is 9·84744. But the content of any sphere divided by $\frac{1}{6}$ of the diameter gives the surface; therefore $500 \div 1{\cdot}64124 = 304{\cdot}6477$ THE SURFACE OF THE SPHERE.

For the cylinder. As 3 : 2 : : 500 : $333\frac{1}{3}$ the solidity of the inscribed sphere (*Theor.* xxvi. *def. page* 45, *vol.* iii.). And $333\frac{1}{3} \div {\cdot}5236 = 636{\cdot}618$ = the cube of the diameter, whence the diameter = 8·6025. And $74{\cdot}0035^* \times {\cdot}7854 \times 6 = 348{\cdot}7340934$ THE SURFACE OF THE CYLINDER.

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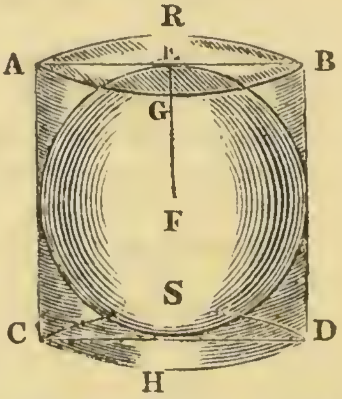
* The square of 8·6025.

(Key to Vol. III. page 53.)

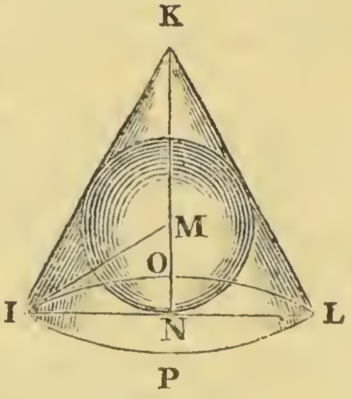
Lastly, for the cone. As 9 : 4 : : 500 : $222\frac{1}{5}$ the solidity of the inscribed sphere; and $222\frac{1}{5} \div {\cdot}5236 = 424{\cdot}412$ the cube of the diameter, from which the diameter is obtained 7·515. But $222\frac{1}{5} \div 7{\cdot}515 \times \frac{1}{6} = 177{\cdot}423$ nearly=the surface of the sphere; consequently it will be, As 4 : 9 : : 177·423 : 399·2017 THE SURFACE OF THE EQUILATERAL CONE. Now $7{\cdot}515 \times \sqrt{3} = 13{\cdot}016361762$ the diameter of the base, and length of the slant line. *Which were to be determined.*

Ex. 13. *For the circumscribing cylinder.* Because, As 1·000 : 3·1415926535 &c. so is the diameter of a circle, to the circumference; and that, radius $\times \frac{1}{2}$ circumference = the area = ·7853981634 nearly; and because (*it is prov'd page* 49, *vol.* ii.) that, the superficies of the sphere is equal to the curve surface of its circumscribing cylinder=diameter × circumference = 3·1415926535 = 4 times the area of the base=4 times the area of a great circle of the sphere, it follows that the whole area of the cylinder ABCD is equal to 4 great circles + the circle CSDH + the opposite circle ARBG; that is, equal to 6 great circles of the inscribed sphere. Therefore the cylinder's convex and whole surface are, respectively as 4 and 6, to the surface of the inscribed sphere, whose superficies is 4.



For the circumscribing equilateral cone. Because, when MN=1, then IN=$\sqrt{3}$, (*since, As Rad.* (MN) : 1 : : *Tang.* 60° ($\angle$ IMN) : $\sqrt{3}$, *and the doubles in the same ratio*), it is evident that the circumference IOLP= $3{\cdot}1415926535897 \times \sqrt{3}$; and, IK being by the question equal to IL, the convex surface of the cone is truly expressed by $\frac{1}{2}\sqrt{3} \times 3{\cdot}1415926535897 \times \sqrt{3} = \frac{1}{2} \times 3{\cdot}1415926535 \times \sqrt{3} \times \sqrt{3} = \frac{1}{2} \times 3{\cdot}1415926535 \times 3 = 1\frac{1}{2} \times 3{\cdot}1415$ &c.$=4{\cdot}71238898025$. But the area of the circle IOLP (*or, the area of the cone's base to rad.* $\frac{1}{2}\sqrt{3}$) is $\sqrt{3} \times \sqrt{3} \times {\cdot}7853981633$ &c.$=3 \times {\cdot}78539$ &c.$=2{\cdot}356194490125$; which is to the convex surface of the cone as 1 to 2; or, as 3 to 6. Lastly, the surface of a globe, whose diameter is unity, being, as above, 3·1415926535=the surface of the sphere inscribed in the equilateral cone IKL, it is, As 3·1415926535 : 4·7123889025 : : 4 : 6. And, As 3·1415926535 : 4·7123889025+2·356194490125 : : 4 : 9.

Which was to be shown.



(Key to Vol. III. page 53.)

Ex. 14. Since, in the last example, the surfaces, of a sphere, its circumscribing cylinder, and circumscribing equilateral cone are demonstrated to be in the ratio of the numbers 4, 6, and 9, it is plain, by Theor. xxxvi. Isoperimetry, that their solidities are in the same ratio. *Which was required to be proved.*

ANALYTICAL PLANE TRIGONOMETRY.

(Page 76.)

Ex. 7. Because sin. $18° = \text{co. sin. } 72° = \frac{1}{2}$ the side of a decagon inscribed in a circle $= \frac{1}{2}\sqrt{\frac{5}{4}} - \frac{1}{4}$, if the surd be reduced to simpler terms by $\times^{\text{lying}}$ the denominator of $\frac{1}{2}$ by 4, and the numerator of the surd by 4 in the form of the square root $(=16,)$ there arises $\frac{1}{2}\sqrt{\frac{5}{4}} = \frac{1}{8}\sqrt{\frac{80}{4}} = \frac{1}{8}\sqrt{20} = \frac{1}{8}\sqrt{4}\sqrt{5} = \frac{1}{8} \times 2 \times \sqrt{5} = \frac{1}{4}\sqrt{5}$; therefore $\frac{1}{2}\sqrt{\frac{5}{4}} - \frac{1}{4} = -\frac{1}{4} + \frac{1}{4}\sqrt{5}$, which $\times^{\text{ed}}$ by the rad. $=$ rad. $\times (-\frac{1}{4} + \frac{1}{4}\sqrt{5}) = \frac{1}{4}$ rad. $\times (-1 + \sqrt{5})$. And so for sin. $54° = \text{co. sin. } 36°$ of which the equation is $\frac{1}{2}\sqrt{\frac{5}{4}} + \frac{1}{4}$ &c. Q. E. D.

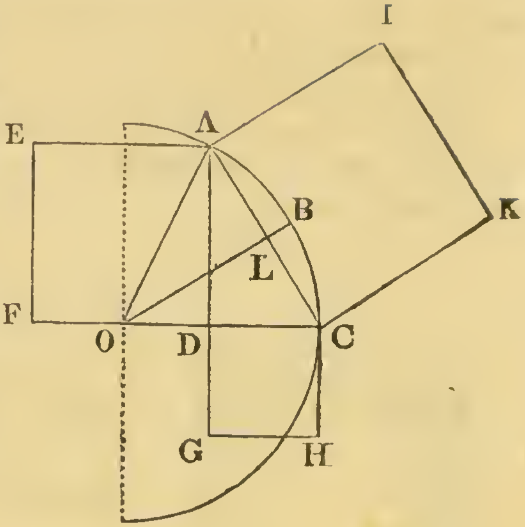
Ex. 8. If P and Q be any two arcs whatever, since sin. $(P+Q) =$ sin. $(P-Q) + 2$ cos. $P \times$ sin. Q, it is evident that when $P = 60°$, then cos. $P =$ cos. $60° =$ half the radius $= \frac{1}{2}$, and the sin. $(60° + Q) =$ sin. $Q +$ sin. $(60° - Q)$. Q. E. D.

Ex. 9. *First.* Let A, and $60° - A$, be the two arcs, then $\frac{1}{2}$ their diff. is $A - 30$. But, As sin. $(A - 30°)$: sin. $A -$ sin. $(60° - A)$ $=$ sin. $A \times$ cos. $30° -$ sin. $30° \times$ cos. A : : sin. $A -$ sin. $60° \times$ cos. $A +$ sin. $A \times$ cos. $60° =$ sin. $A \times$ cos. $30° -$ sin. $30° \times$ cos. A : $\frac{3}{2}$ sin. $A -$ sin. $60° \times$ cos. $A = \frac{1}{2}\sqrt{3} \times$ sin. $A - \frac{1}{2}$ cos. A : $\frac{3}{2}$ sin. $A - \frac{1}{2}\sqrt{3} \times$ cos. $A \because 1 : \sqrt{3}$.

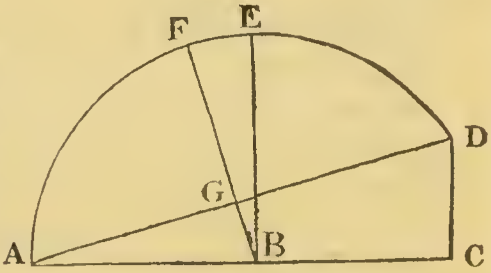
Next. Let A and $90° - A$ be the two arcs; then $\frac{1}{2}$ their difference is $A - 45°$. But,

As sin. $(A - 45°)$: sin. $A -$ sin. $(90 - A) =$ sin. $A \times$ cos. $45° -$ sin. $45° \times$ cos. A : : sin. $A -$ cos. $A = \dfrac{1}{\sqrt{2}} : 1 \because 1 : \sqrt{2}$. Q.E.D.

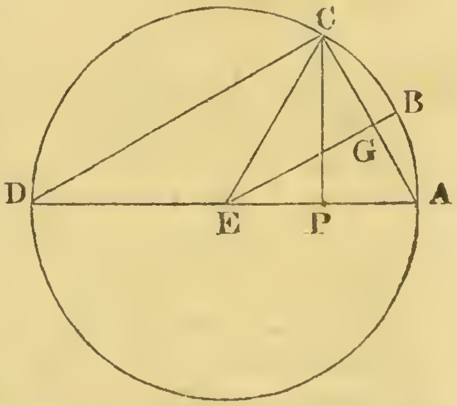
(Key to Vol. III. page 76.)



Ex. 10. Let ABC be any arc whatever, bisected in the point B; then AD is the sine of the arc ABC, and DC is its versed sine. But the angle ADC is a right angle, and the square AEFD, together with the square CDGH, equal to the square AEKC (*Euc.* i. 47) Now AC the chord of the arc ABC is bisected at L, and AL (=LC) is the sine of the arc AB = BC = $\frac{1}{2}$ ABC, *therefore* &c. Q. E. D.

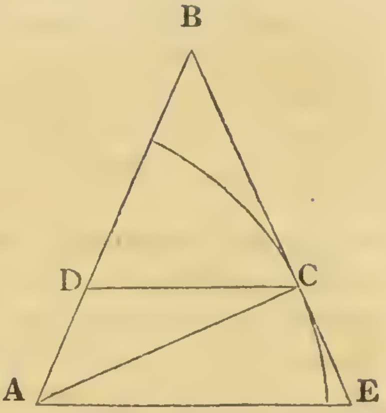


Ex. 11. If AF be any arc whatever, and AFD its double, then is AG the sine of the mean arc, AD twice the sine of the mean arc, AC the versed sine of double the arc AF, and AB the rad. of the circle. But, As AB : AG : : AD : AC, that is, As rad. : sin. mean arc. : : twice sin. mean arc : vers. sin. double arc; or, putting a for the radius, b for the sine of the arc AF, and c for the half of AC, it will be, As $a : b :: 2b : 2c$, wherefore $\frac{1}{2}a : b :: b : 2c$. Hence "*the sine of an arc,*" &c. Q. E. D.



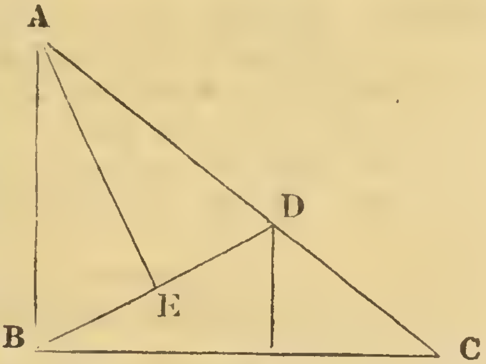
Otherwise. Let AB be an arc of a circle, ABC its double, and AGC the chord of the double arc; then DCA being an angle in a semicircle is a right angle, and AGC is a mean proportional between the diameter AD and the segment of the base AP. (*Eucl.* vi. 8.) But $4AG^2$ ($=4\sin^2$. of the arc AB) $= AGC^2 =$ AD × AP = 2AE × AP. That is, $AG^2 = \frac{1}{2}AE \times AP$. Wherefore AG (=sin. arc AB,) is a mean proportional between half the radius and AP (the versed sine of double the arc.) Q. E. D.

(Key to Vol. III. page 76.)

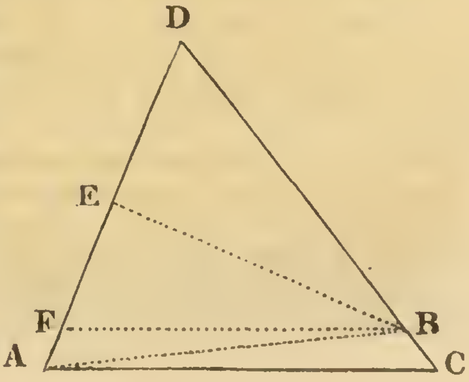


Ex. 12. Let BAC be any angle whatever, AB its secant, and CB its tangent. In BA take BD equal to BC, and join CD; then parallel to DC draw AE, and produce BC to E. Because BD and BC are equal, their opposite angles are equal: but the angle BDC is evidently equal to BAC together with half its complement; therefore the angle BEA is equal to the angle BAC together with half its complement. Whence it follows, that the angle CAE is half the complement of BAC. Now CE is the tangent of the angle CAE, and equal to AD: wherefore EB is equal to AB; that is, the secant of BAC is equal to the tangent of the same angle, together with the tangent of half its complement. Q. E. D.

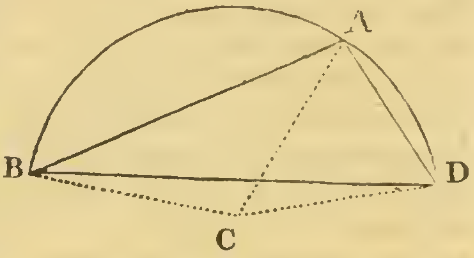
*_** This problem may likewise be demonstrated by the algebraical expressions given in the III. chap. of Hutton's 3d volume, or in WOODHOUSE's elegant *Treatise on Plane and Spherical Trigonometry*, Cambridge, 1813.



Ex. 13. Let ABC be the triangle, whereof BC is the base; and since by the hypothesis AB is unequal to AC, AD may be supposed equal to AB. Join BD, and from A draw AE at right angles to BD. Then, because ABD is isoceles, and the vertical ∠ at A the same in the two triangles ABD and ABC, it is evident that the sum of the angles at the base of the one triangle is equal to that of those at the base of the other; and that the ∠ ABD is half the sum of the $\angle^{les}$ ABC, ACB; but the ∠ CBD is the excess of the ∠ ABC above that half sum, consequently equal to half their difference. Now, As BC : DC (the difference of the sides) : : sin. ∠ ADB (= ∠ ABD) : sin. ∠ DBC.



Again, for the second Theorem of the Example. Let ABC be the given triangle, and produce CB indefinitely; also make BD=AB. Join AD, and from B draw BE perpendicular to AD, and BF parallel to AC. Then it is manifest that the ∠ BDA = ∠ DAB = $\frac{1}{2}$ ∠ ABC. It is likewise evident that the ∠ DBA (external ∠) is equal to both the angles at the base of the triangle ABC, therefore the $\angle$ ABE equal to half the sum of $\angle^{\text{les}}$ BAC, ACB; and EBF being the excess of the greater above that half sum, is equal to half their difference. But, As AC : CD (sum of the sides) : : co. sin. $\angle$ DBE : co. sin. EBF. *Which was to be proved.*



Ex. 14. Because the three arcs constitute an entire circumference, and are to one another as 1, 2, 4, it follows that their chords (see the accompanying figure, wherein the three chords are DB, BA, and AD) are 177·9, 142·76, 79·24. This question may likewise be solved by the common rules of Trigonometry, since it only requires a triangle of which the perimeter shall be 400, and the angles $25°\frac{5}{7}$, $51°\frac{3}{7}$, and $102°\frac{6}{7}$; for the sines of the angles being as the sides which subtend them, it is evident that AD$=$79·239 &c. BD$=$177·924 &c. and AB $=$ 142·756. *Which was required to be shown.*

Ex. 15. Because (page 57, vol. iii.) "*The sides of plane triangles are proportional to the sines of their opposite angles,* and *vice versâ,*" the sides must also be as 17, 15, and 8. But 160 divided into three parts in that ratio, gives 68, 60, and 32 for the sides of the triangle. Consequently the angles are 89° .. 58′, 61° .. 57′, and 28° .. 5′. *Which was required.*

Ex. 16. The greater logarithm = 2·5378191
The less logarithm = 2·2407293

The difference of the log^{s}. is 0·2970898

Adding 10 to the index* of this difference, the result is the log. tangent of 63° .. 13′, from which when 45° have been taken, the remainder is 18° .. 13′, of which the log. tang.$=$9·5173353. And the log. co. tang. of 18° .. 40′ (half the given angle)$=$10·4712979, of which the sum (abating 10 in the index for the 10 added to the first difference) is 9·9886332.$=$log. tang. of 44° .. 15′, therefore 115° .. 35′ and 27° .. 5′ are the angles required. (*See vol.* iii. *page* 70.)

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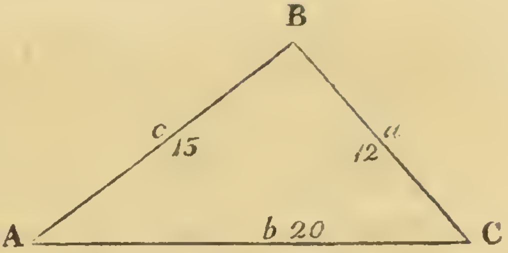
* *It will be observed, that in many of our solutions we scrupulously follow Dr. Hutton's Rules, whereas in others we abbreviate the operation by means not mentioned in the* COURSE. *In this Example the addition of* 10 *to the index of the log. difference and the subsequent abatement of* 10 *in the log. sum of the tangent and co-tangent might be avoided, and less liability to error ensue, if the log. tangents, &c. to radius* 1, *were used. For this use see various parts of the* KEY.

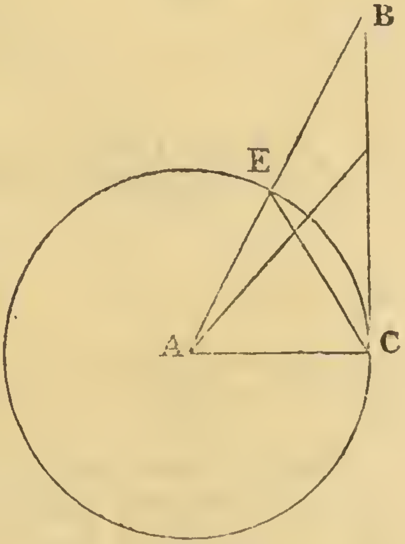
(Key to Vol. III. page 76.)



Ex. 17. The fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, being to one another as the numbers 20, 15, and 12, if these be assumed as the sides of the triangle, the angles by the *second* formula (page 56, vol. iii.) are, A$=36°\,..\,43'$, since $\frac{400+225-144}{2\times 15\times 20}=\cdot 80166\dot{6}=$natural co. sin. 36° .. 43′.

B$=94°\,..\,56'$, because $\frac{144+225-400}{2\times 12\times 15}=-\cdot 08611\dot{1}=$ nat. co. sin. 94° .. 56′. and C$=48°\,..\,21'$ since $\frac{144+400-225}{2\times 12\times 20}=\cdot 664583=$ nat. co. sin. 48° .. 21′, *nearly*. But by formula xxxiv. sin. $\frac{1}{2}\angle$ A$=\sqrt{\frac{(\frac{1}{2}s-b)\cdot(\frac{1}{2}s-c)}{bc}}=\sqrt{\frac{3\cdot 5\times 8\cdot 5}{300}}=\cdot 3149073=$sin. $18°\,..\,21'\,..\,23''$, therefore A$=36°\,..\,42'\,..\,46''$.

And sin. $\frac{1}{2}\angle$ B $=\sqrt{\frac{(\frac{1}{2}s-a)\cdot(\frac{1}{2}s-c)}{ac}}=\sqrt{\frac{11\cdot 5\times 8\cdot 5}{180}}=\cdot 736925$ $=$sin. $47°\,..\,28'\,..\,1''$, therefore B $=94°\,..\,56'\,..\,2''$, consequently C $=48°\,..\,21'\,..\,12''$. Q. E. I.



Ex. 18. Let AB be the secant of 60° and CD the tangent of 45°; and join CE. Then because DAC is an angle of 45°, and ACD a right angle, the angle CDA is likewise 45°, and CD $=$ AC $=$ Rad. Again because EAC is an angle of 60° and AE $=$ AC, the triangle AEC is equilateral, and CE $=$ Radius. But the angle at B$=30°$, and the angle BCE $=$ compl. 60° $=$ 30°; therefore EB$=$EC$=$Rad.$=$DC. Now AE $=$ EB, that is, AB $=$ twice DC. Moreover it is manifest that the square of AD (that is, of the secant of 45°) $=2r^2$, and AB ($=2r=$secant of 60°) multiplied by CD ($=r=$tang. 45°) $=2r^2$. Wherefore AD is a mean proportional between CD and AB. Q. E. D.

☞ *This problem (like most of the others which we prove geometrically) admits of easy demonstration by means of the algebraic expressions for the secant, tangent, &c. Whoever desires to see these subjects treated in a manner as perspicuous as elegant, is referred to Woodhouse's Trigonometry, 2nd edition.*

(Key to Vol. III. page 77.)

Ex. 19. Because the rectangle of the sines of any two arcs is equal to the rectangle of half the radius and the difference of the co-sines of the sum and difference of the two arcs$=\frac{1}{2}$ sum of the chords $\times\ \frac{1}{2}$ difference of the chords of the two arcs$=\frac{1}{4}sd$, (where s is the sum and d the difference of the chords,) it follows that 4 times the rectangle of the sines of the two arcs$=sd$. But the difference of the squares of the chords is equal to a rectangle under the sum and difference of those chords $=sd$; therefore "4 times the rectangle &c." Q. E. D.

Otherwise.

Let A and B represent any two unequal arcs, whereof A is the greater; then, since $\sin.(P+Q)\times\sin.(P-Q)=\sin^2.P-\sin^2.Q$, put $P+Q$ for A, and $P-Q$ for B, so shall $P=\frac{1}{2}(A+B)$, and $Q=\frac{1}{2}(A-B)$; and $\sin.A.\times\sin.B=\sin^2.\frac{1}{2}(A+B)-\sin^2.\frac{1}{2}(A-B)$.

Consequently $4(\sin.A\times\sin.B.)=4\times\sin^2.\frac{1}{2}(A+B)-4\times\sin^2.\frac{1}{2}(A-B)=\text{chord}^2$ of $\frac{1}{2}(A+B)-\text{chord}^2$ of $\frac{1}{2}(A-B)$. *Which was to be proved.*

Ex. 20.

$$\sqrt{\frac{(\frac{1}{2}s-b)\cdot(\frac{1}{2}s-c)}{bc}}=\sqrt{\frac{\overline{\frac{5+6+7}{2}-6}\times\overline{\frac{5+6+7}{2}-7}}{6\times7}}=\sqrt{\tfrac{1}{7}}.$$

Also,

$$\sqrt{\frac{(\frac{1}{2}s-a)\cdot(\frac{1}{2}s-c)}{ac}}=\sqrt{\frac{\overline{\frac{5+6+7}{2}-5}\times\overline{\frac{5+6+7}{2}-7}}{5\times7}}=\sqrt{\tfrac{8}{35}}.$$

And,

$$\sqrt{\frac{(\frac{1}{2}s-a)\cdot(\frac{1}{2}s-b)}{ab}}=\sqrt{\frac{\overline{\frac{5+6+7}{2}-5}\times\overline{\frac{5+6+7}{2}-6}}{5\times6}}=\sqrt{\tfrac{2}{5}}.$$

Now* $10+\text{log. of }\sqrt{\tfrac{1}{7}}=9{\cdot}5774504=\text{log. sin. }22^\circ..\ 12'..\ 27''$.
$10+\text{log. of }\sqrt{\tfrac{8}{35}}=9{\cdot}6795106=\text{log. sin. }28^\circ..\ 33'..\ 42''$.
$10+\text{log. of }\sqrt{\tfrac{2}{5}}=9{\cdot}8010299=\text{log. sin. }39^\circ..\ 13'..\ 51''$.

Wherefore,

$$\left.\begin{aligned}\angle A&=2(22^\circ..12'..27'')=44^\circ..24'..54''\\ \angle B&=2(28^\circ..33'..42'')=57^\circ..\ 7'..24''\\ \text{And }\angle C&=2(39^\circ..13'..51'')=78^\circ..27'..42''\end{aligned}\right\}\text{Ans.}$$

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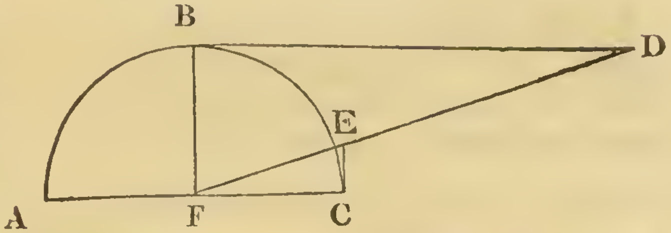
* *See page 74, vol.* iii.

(Key to Vol. III. page 77.)



Ex. 21. Let ABC be a semi-circle, BD the tangent of the arc BE, and EC its co-tangent. Then because $\frac{rad.^2}{tang.}$ = *co-tang.*, if x be put for the length of the tangent (BD) and a for the radius of the circle, then $\frac{a^2}{x}=$ the co. tangent (EC). But by the question $4a=x+\frac{a^2}{x}$; that is, $x^2-4ax=-a^2$. Hence $x^2-4ax+4a^2=3a^2$; and $x=2a\pm\sqrt{3a^2}=2a\pm a\sqrt{3}$. Now when radius is unity this equation becomes $x=2\pm\sqrt{3}=3·7320508$ or ·2679492, of which the logarithms are ·5719475 and −1·4280525 = the log. tangents* of 75° and 15°, either of which is the arc required.

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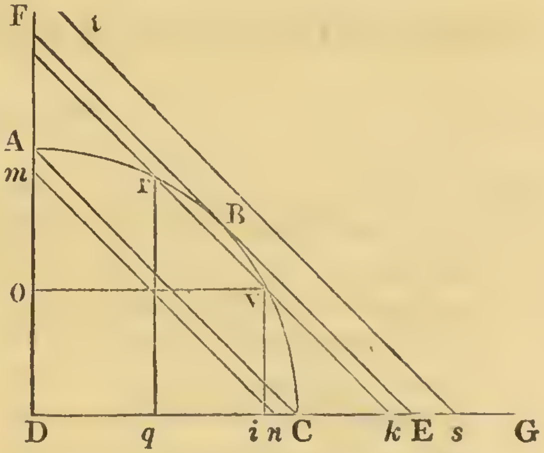
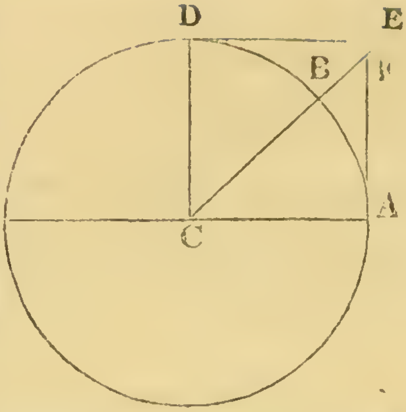
* *To radius* 1. *If* 10 *be added to the index, the log. sum will be the common log. tangent.*



Ex. 22. Let ABC be a quadrant of a circle, of which the center is D; and let the radii DA and DC be produced indefinitely towards F and G. Then in DG take Dk equal to a, and in DF take Dl equal to Dk. Join kl, so shall kl, if it cut or touch the quadrant, indicate two arcs either of which will answer the conditions. Ar, therefore, is an arc of which the sine added to the cosine is equal to a, that is, equal to Dk; which is evident, since Dq is equal to the sine, and $qr=qk$, equal to the cosine. Or, Av, is the arc, in which case $ov=$ Di is the sine, and $vi=ik$, the cosine; together equal to D$k=a$.

By which it is manifest, that the limits of possibility are double the sine of 45° on the one hand, and the sine of 90° on the other; for when a exceeds the former limit, and is equal to Ds, then $s\,t$ neither cuts nor touches the quadrant; and when a falls short of the latter limit and is equal to Dn, the line nm cannot meet the curve. Consequently DC is the least, and DE the greatest possible value of a. Q. E. I.

(Key to Vol. III. page 77.)



Ex. 23. If AB represent the arc sought, AF its tangent, CF its secant, and DE its co-tangent, then by the question CF=DE. But the co-tang. $= \frac{rad.^2}{tang.}$, and sec. $= \sqrt{rad.^2 + tang.^2}$ Putting, therefore, x for AF (the tangent of the arc required) the equation is $\frac{rad.^2}{x} = \sqrt{rad.^2 + x^2}$, that is, (if radius be unity,) $1 = x^2 + x^4$, whence, by completing the square of this quadratic, $x^4 + x^2 + \frac{1}{4} = \frac{5}{4}$, and $x^2 = -\frac{1}{2} \pm \sqrt{\frac{5}{4}}$, consequently $x = \sqrt{-\frac{1}{2} \pm \sqrt{\frac{5}{4}}} = \cdot786151$ of which the log. is ·8955060=log. tangent of 38° .. 10′ .. 21″ .. 42‴=the arc AB. Q. E. I.

RESOLUTION OF SPHERICAL TRIANGLES.

(Page 108.)

Ex. 2. Tang. c = tang. a × co. sin. B=log. tang. a + log. co. sin. B − 10.=10·6851149+9·9469372−10·=10·6320521=log. tangent of 76° .. 52′=the side c. And sin. b=sin. a × sin. B =log. sin. A + log. sin. B − 10 = 9·9909338+9·6680265−10 =9·6589603=log. sin. of 27° .. 8′=the side b. And tang. C =co. tang. B÷co. sin. A = log. cot. B − log. co. sin. a+10= 10·2789107−9·3058189 + 10= 10·9730918=log. tang. of 83° .. 56′=the angle C. Ans.

Ex. 3. Tang. a=tang. b÷co. sin. C = log. tang. 117° .. 34′−log. co. sin. of 31°·51′, + 10 = 10·2822906−9·9291289+10· = 10·3531617 = the log. tang. of 66° .. 5′ or 113° .. 55′, which last=a=the hypothenuse. And cos. B=co. sin. b × sin. C= log. cos. 117° .. 34′ + log. sin. 31° .. 51′, −10. = 9·6653749 + 9·7223848 − 10. = 9·3877597 =log. co. sin. of 75° .. 52′, or 104° .. 8′, whereof the latter = B, the third angle of the triangle.

Again, tang. c=sin. b × tang. C=log. sin. 117° .. 34′+log. tang. 31° .. 51′ − 10. = 9·9476655 + 9·7932560 − 10. = tang. 28° .. 51′ = C, the third and last side sought.

(Key to Vol. III. page 108.)

Ex. 4. Co. sin. a = co. sin. $b \times$ co. sin. c = log. co. sin. 27° .. 6′ + log. co. sin. 76° .. 52′ − 10. = 9·9494938 + 9·3564426 − 10 = 9·3059364 = log. co. sin. 78° .. 20′ = a = hypothenúse. Also, Tang. B = tang. b ÷ sin. c = log. tang. 27° .. 6′ − log. sin. 76° .. 52′ + 10 = 9·7090374 − 9·9884894 + 10 = 9·7205480 = log. tang. 27° .. 44′ = the angle B. And, tang. C = tang. c ÷ sin. b. = log. tang. 76° .. 52′ − log. sin. 27° .. 6′ + 10. = 10·6320468 − 9 6585312 + 10 = 10 9735156 = log. tang. 83° .. 56 = the angle C. *Which was to be done.*

Ex. 5. Sin. a = sin. b ÷ sin. B, hence, −1·8271887 = log. sin. b.
−1·8710735 = log. sin. B.

log. diff. −1·9561152 = log. sin. of an ∠ of 64° .. 40′½, or of its supplement = a = the hypothenuse. And

Sin. c = tang. b ÷ tang. B, hence −1·9574850 = log. tang. b.
0 0455626 = log. tang. B.

log. diff. −1·9119224 = log. sin. of an ∠ of 54° .. 44′, or of its supplement = the side c. Lastly,

Sin. C = co. sin. B ÷ co. sin. b, hence, −1·8255109 = log. cos. B.
−1·8697037 = log. co. sin. b.

log. diff. −1·9558072 = log. sin. of an ∠ of 64° .. 35′, or of its supplement = the angle C.

Ex. 6. Given B = 48°, C = 64° .. 35′, and A a right angle. Now co. sin. a = cot. B × cot. C. But

Log. cot. 48° .. 0′ is −1·9544374
Log. cot. 64° .. 35′ is −1·6768686

Log. cos. 64° .. 40′½ = a, is −1·6313060 log. sum.

And cos. b = cos. B ÷ sin. C. Therefore

Log. cos. 48° is −1·8255109
Log. cos. 64° .. 35′ is −1·9557890

Log. cos. 42° .. 12′ = b, is −1·8697219 log. diff.

Likewise cos. c = cos. C ÷ sin. B. Hence,

Log. cos. 64° .. 35′ being −1·6326576
Log. sin. 48° .. 0′ being −1·8710735

Log. cos. 54° .. 44′ = c, is −1·7615841 log. sum.

Therefore a = 64° .. 40′½, b = 42° .. 12′, and c = 54° .. 44′. Ans.

(Key to Vol. III. page 109.)

Ex. 7. Taking the *quadrantal* side a as radius; 115° .. 20′ the supplement of the ∠A, as hypothenuse; and 42° .. 12′ the ∠C as a side; it will be (changing *like* for *unlike* in the determinations,) by Table I. page 102,

$$\text{Sin. } c = \frac{\text{sin. } 42° \,..\, 12'}{\text{sin. } 115° \,..\, 20'}.$$

$$\text{Cos. } b = \frac{\text{tang. } 42° \,..\, 12'}{\text{tang. } 115° \,..\, 20'}.$$

$$\text{And} \quad \text{Cos. B} = \frac{\text{cos. } 115° \,..\, 20'}{\text{cos. } 42° \,..\, 12'}.$$

By Logarithms.

Log. sin. 42° .. 12′ = −1·8271887
Log. ar. co. of sin. 115° .. 20′ = 0·0439114

Log. sum − 1·8711001*
Add 10 to the index 10·

Sin. c=sin. 48° .. 0′ .. 13″½. *Log.* 9·8711001

* −1·8711001 is the log. sin. 48° .. 0′ .. 13″½ to radius 1.

And for the cos. b it is,

Tang. 42° .. 12′ - - - - - - Log. −1·9574850
Tang. 115° .. 20′ - - - Log. ar. co. −1·6752372

Cos. b=*cos.* 115° .. 25′ .. 14″½ *Log.* −1·6327222

Lastly for the cos. A it will be,

Cos. 115° .. 20′ - - - - - - - - Log. −1·6313258
Cos. 42° .. 12′ - - - - Log. ar. co. 0·1302963

Cos. B=*cos.* 125° .. 16′ .. 53″ *Log.* −1·7616221

$$\text{Wherefore} \left.\begin{array}{l} \text{B} = 125° \,..\, 16' \,..\, 53'' \\ b = 115° \,..\, 25' \,..\, 14''\tfrac{1}{2} \\ c = 48° \,..\, 0' \,..\, 13''\tfrac{1}{2} \end{array}\right\} \textit{Which were required.}$$

(Key to Vol. III. page 110.)

Ex. 9. Given $c = 114° .. 30′$, $a = 56° .. 40′$, and $C = 125° .. 20′$. Hence it is by logarithms, (using the common indices,)

As Sin. 114° .. 30′ (c) - Log. ar. co. –10·0409771
: Sin. 125° .. 20′ (C) - - - - - Log. 9·9115844
: : Sin. 56° .. 40′ (a) - - - - - Log. 9·9219401

: *Sin.* 48° .. 30′ (A) - - - - - *Log.* 9·8745016

And

$$\text{Co. tang. } \tfrac{1}{2} \angle B = \frac{\text{Tang. } \tfrac{1}{2} \text{ diff. of A and C} \times \text{sin. } \tfrac{1}{2} \text{ sum of } a \text{ and } c.}{\text{Sin. } \tfrac{1}{2} \text{ diff. of } a \text{ and } c.}$$

But ½ diff. of A and C=38° .. 25′, ½ sum of a and c=85° .. 35′, and ½ diff. of a and c=28° .. 55′; wherefore

+ Log. tang. 38° .. 25′ = 9·8993082
+ Log. sin. 85° .. 35′ = 9·9987084
– Log. sin. 28° .. 55′ ar. co. = –10·3255703

Log. sum - - - - - - - - - - 10·2135869 = Log. co. tang. 31° .. 27′½, which ∠ doubled = 62° .. 55′ = ∠B. But

As Sin. 48° .. 30′ (A) - - Log. ar. co. –10·1255439
: Sin. 56° .. 40′ (a) - - - - - - - Log. 9·9219401
: : Sin. 62° .. 55′ (B) - - - - - - - Log. 9·9495585

: *Sin.* 83° .. 19′ (b) - - - - - - - *Log.* 9·9970425

Whereby the Answer is obvious.

Ex. 10. Given A=48° .. 30′ C=125° .. 20′, and c (it was intended) to be given) = 114° .. 30′. Therefore it will be

As Sin. 125° .. 20′ (C) - - Log. ar. co. – 10·0884156
: Sin. 114° .. 30′ (c) - - - - - - - Log. 9·9590229
: : Sin. 48° .. 30′ (A) - - - - - - - - Log. 9·8745016

: *Sin.* 56° .. 40′ (a) - - - - - - - *Log.* 9·9219401

$$\text{And Tang. } \tfrac{1}{2} b = \frac{\text{Tang. } \tfrac{1}{2} \text{ diff. of } a \text{ and } c \times \text{sin. } \tfrac{1}{2} \text{ sum of A and C.}}{\text{Sin. } \tfrac{1}{2} \text{ diff. A and C.}}$$

But ½ diff. of a and c=28° .. 55′, ½ sum of A and C=86° .. 55′, and ½ diff. of A and C=38° .. 25′. Consequently it is

+ Log. tang. 28°..55′=9·7422609
+ Log. sin. 86°..55′=9·9993640
19·7416249
Subtract log. sin. 38°..25′= 9·7933543

Log. tang. 41°..36′ = 9·9482706 diff.
Multiply by 2

Product 83°..12′ = the side b.

Again, As Sin. (a) 56°..40′ Log. ar. co. –10·0779768
: Sin. (A) 48°..30′ - - - Log. 9·8744561
: : Sin. (b) 83°..12′ - - - Log. 9·9969342

: *Sin.* (B) 62°..55′ - - - *Log.* 9·9493671

Ex. 11. Given c=114°..30′, a=56°..40′, and B=62°..54′.

Now,

$$\text{Tang. } \tfrac{1}{2} \text{ diff. of A and C} = \frac{\text{Co. tang. } \frac{1}{2}\text{B} \times \text{sin. } \frac{1}{2} \text{ diff. of } a \text{ and } c.}{\text{Sin. } \frac{1}{2} \text{ sum of } a \text{ and } c.}$$

And

$$\text{Tang. } \tfrac{1}{2} \text{ sum of A and C} = \frac{\text{Co. tang. } \frac{1}{2}\text{B} \times \text{co. sin. } \frac{1}{2} \text{ diff. of } a \text{ and } c.}{\text{Co. sin. } \frac{1}{2} \text{ sum of } a \text{ and } c.}$$

But $\frac{1}{2}$B=31°..27′, $\frac{1}{2}$ diff. of a and c=28°..55′, and $\frac{1}{2}$ sum of a and c=85°..35′. Therefore it is

+ Log. co. tang. 31°..27′ = 10·2132480
+ Log. sin. 28°..55′ = 9·6844297
19·8976777
– Log. sin. 85°..35′ = - - - - - - 9·9987084

Log. tang. 38°..24′ = 9·8989693 diff.

And,

+ Log. co. tang. $\frac{1}{2}$B=31°..27′=10·2132480
+ Log. co. sin. 28°..55′= 9·9421688
20·1554168
– Log. co. sin. 85°..35′= - - - - - 8·8865418

Log. tang. 86°..55′ = 11·2688750 diff.

Consequently A=48°..30′, and C=125°..20′.

Lastly, As Sin. (A) 48° .. 30′ Log. ar. co. − 10·1255439
: Sin. (*a*) 56° .. 40′ - - - - Log. 9·9219401
: : Sin. (B) 62° .. 54′ - - - - Log. 9·9494938

: *Sin.* (*b*) 83° .. 15′ - - - - *Log.* 9·9969778

Ex. 12. Given C 125° .. 20′, *b* 83° .. 12′, and A 48° .. 30′.

$$\text{Ans.}\quad \left.\begin{array}{rl} a = & 56^\circ\,..\,40' \\ c = & 114^\circ\,..\,30' \\ \text{and B} = & 62^\circ\,..\,54' \end{array}\right\}\ \textit{Which were required.}$$

Ex. 13. Given A = 48° .. 31′, B = 62° .. 56′, and C = 125° .. 20′.

$$\left.\begin{array}{rl} a = & 56^\circ\,..\,40' \\ b = & 83^\circ\,..\,12' \\ c = & 114^\circ\,..\,30' \end{array}\right\}\quad \text{Ans.}$$

Ex. 14. Given A = 50° .. 12′, C = 58° .. 8′, and *a* = 62° .. 42′.

Wherefore it will be,

As Sin. (A) 50° .. 12′ - - - - Log. ar. co. − 10·1144385
: Sin. (*a*) 62° .. 42′ - - - - - - - - - Log. 9·9487147
: : Sin. (*c*) 58° .. 8′ - - - - - - - - - Log. 9·9290504

: *Sin.* (*c*) 79° .. 11′, *or its suppl.* 100° .. 49′, *Log.* 9·9922036

$$\text{Now, co. tang. } \tfrac{1}{2}\,\text{B} = \frac{\text{Tang. } \tfrac{1}{2} \text{ diff. A and C} \times \text{sin. } \tfrac{1}{2} \text{ sum } a \text{ and } c}{\text{Sin. } \tfrac{1}{2} \text{ diff. } a \text{ and } c}$$

But ½ diff. A and C = 3° .. 58′, ½ sum *a* and *c* = 70° .. 56′½, and ½ diff. *a* and *c* = 8° .. 15′.

Consequently

$$\text{Add}\left\{\begin{array}{ll} \text{log. tang.} & 3^\circ\,..\,58' = 8{\cdot}8409977 \\ \text{log. sin.} & 70^\circ\,..\,56'\tfrac{1}{2} = 9{\cdot}9755175 \end{array}\right.$$

18·8165152
and subtract log. sin. 8° .. 15′ = 9·1568296

Log. co. tang. 65° .. 26′ = 9·6596856 diff.
Multiply by 2

PRODUCT = ∠B = 130° .. 52′, or, by taking the other value of *c* (= 100° .. 49′), ∠B = 156° .. 17′; hence, by common analogy,

(Key to Vol. III. page 110.)

As Sin. (A) 50° .. 12′ Log. ar. co. −10·1144385
: Sin. (a) 62° .. 42′ - - - - Log. 9·9487147
: : Sin. (B) 130° .. 52′ - - - - Log. 9·8786563

: *Sin.* (b) 119° - - - - - - - *Log.* 9·9418095,

Or taking B = 156° .. 17′, b is 152° .. 17′. Ans.

Ex. 15. Because (*Theor.* v. Spherical Trigonometry.)

As 360° : 1″ : : $\frac{1}{2}$ surface of the earth : area required, it will be, putting $a = 3·14159$ &c. and $d = 7957·75$ the diameter given,

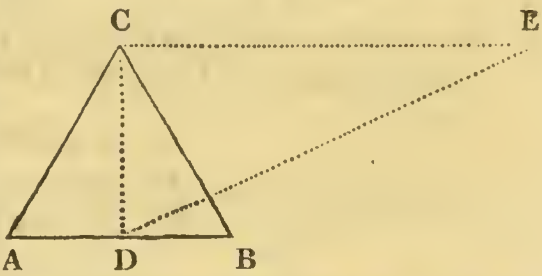
As 1296000″ : 1″ : : $\frac{ad^2}{2}$: *area required;* that is, $\frac{ad^2}{2 \times 1296000}$ $= \frac{ad^2}{2592000}$ = area of the triangle measured on the ⊖'s surface.

Now, $d = 7957·75$, whereof the log. is 3·9007903
Multiply by 2

PRODUCT 7·8015806 = log. d^2
And, $a = 3·141592$, whereof the log. is ·4971500

Sum 8·2987306 = log. ad^2
Lastly, the log. of 2592000 is - - - 6·4136350

Difference 1·8850956 = log. of the area = 76·75299 square miles *and upwards.* Ans.



Ex. 16. *First for the plane angle between the base and each face.* Since by the properties of a hexagon each side of the base = radius, and because by the question, the altitude is double each side of the base, that is, double the radius; it will be

As Sin. 90° - - - Log. ar. co. −10·0000000
: 1 (= CB) - - - - - - - - - Log. 0·0000000
: : Sin. 60° (∠DBC) - - - - - Log. 9·9375306

: CD ·8660254 - - - - - *Log.* −1·9375306

(Key to Vol. III. page 110.)

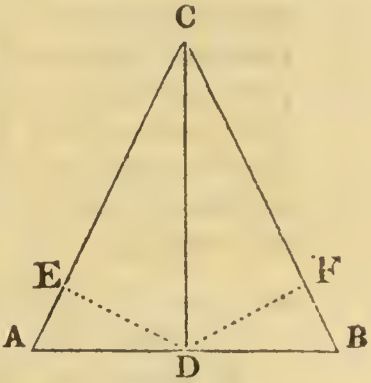
And, As 2 (CE) - - - - - - - - - Log. ar. co. $-1·6989700$
: Rad. (CE) - - - - - - - - - - - Log. $10·0000000$
: : ·8660254 (CD) - - - - - - - - Log. $-1·9375306$

: *Co. tang.* $66° .. 35' .. 11''\frac{1}{2}$ ∠CDE - *Log.* $9·6365006$

Next, for the plane angle between each pair of lateral faces. Because CD is double of AD or of DB, it will be,

As 2 (CD) - Log. ar. co. $-1·6989700$
: 1 (AD) - - - - - Log. $0·0000000$
: : Rad. (CD) - - - Log. $10·0000000$

: *Co. tang.* $63° .. 26' .. 5''\frac{3}{4}$ ∠CAD - - - - *Log.* } $9·6989700$



But if DE and DF be drawn at right angles to AC and CB, the angle FDE is the measure of the inclination of the plane CDB to the plane CDA. Now it is evident that, FDE is bisected by the straight line CD; consequently the angle CDE is half the measure of the angle required. Again the two triangles EDC and DAC are similar, and the angle CAD = ∠CDE. Wherefore the angle CDE = $63° .. 26' .. 5''\frac{3}{4}$ and the whole angle FDE = $126° .. 52' .. 11''\frac{1}{2}$.

And since the magnitude of a trilateral solid angle is measured by the excess of the sum of the three angles made by its bounding planes above 2 right angles, it follows that, As 360° : $\left\{ 2(66°..35'..11''\tfrac{1}{2}) + 126°..52'..11''\tfrac{1}{2} - 180° \right\}$: : 1000 : 222·34143 *either solid angle at the base of the pyramid.* And for the solid angle at the vertex, since its measure is the excess of the sum of the angles of inclination of the several planes forming it, above double the number of those planes multiplying 90°, when from the product 4 right angles have been taken; it follows that,

As 360° : $\left\{ 6(126° .. 52' .. 11''\tfrac{1}{2}) - 720°) \right\}$: : 1000 : 114·497685 *the solid angle at the vertex of the pyramid.*

☞ *The plane angles agree with Dr. Hutton's answer, but the solid angles disagree.*

(Key to Vol. III. page 146.)

GEODESIC OPERATIONS, AND THE FIGURE OF THE EARTH.

Ex. 1. The arc of a great circle passing through the two distant objects will cut the horizon, and give two similar right angled spherical triangles for solution. Now the hypothenuses of these will be in the ratio of their perpendiculars, that is, in the ratio of 1″ to 25′ .. 47″ ∵ 1 : 1547. Hence, (the sum of the hypothenuses being 66° .. 30′ .. 39″,) the hypothenuse of the greater triangle is 66° .. 28′ .. $4''\frac{1}{5}$. and of the less, 2′ .. $34''\frac{4}{5}$. Therefore, [*Case* 1. *Right angled spherical triangles.*]

As Cos. 0° .. 25′ .. 47″ - - Log. ar. co. −10·0000122
: Cos. 66° .. 28′ .. $4''\frac{1}{5}$ - - - - - - - Log. 9·6012600
: : Radius or sin. 90° - - - - - - - - Log. 10· ———

: *Sin.* 66° .. 28′ .. 1″·67 - - - - - - - *Log.* 9·6012722

But $\frac{1}{1547}$ × (66° .. 28′ .. 1″·67) = 0° .. 2′ .. 34″·7
Add 66° .. 28′ .. 1″·67

Sum and Answer* 66° .. 30′ .. 36″ 37

Otherwise,†

First taking the trouble to correct the formula which in its present state is imperfect, it becomes

$$\text{Log. sin. } \tfrac{1}{2}\,c = \tfrac{1}{2}\left\{ 20 + \text{log. sin. } \tfrac{1}{2}(c + h - h') + \text{log. sin. } \tfrac{1}{2}(c + h' - h) - \text{log. cos. } h' - \text{log. cos. } h. \right\}$$

Then c = 66° .. 30′ .. 39″
h = 25′ .. 47″
h' = −1″

Wherefore $\frac{1}{2}(c+h-h')$ = 33° .. 28′ .. $13''\frac{3}{2}$, and
$\frac{1}{2}(c+h'-h)$ = 33° .. 2′ .. $25''\frac{1}{2}$.

---

* Which we submit to be more than half a second nearer the true horizontal angle, than is the Answer given by Dr. Hutton.

† *The reader, if he desires it, will find the figure, and other particulars, in Woodhouse's Trigonometry, 2nd edition.*

(Key to Vol. III. page 146.)

Consequently it is

Radius square* - - - - - - - -	Log.	20· ———	
Sin. 33° .. 28′ .. 13″½ - - - - - -	Log.	9·7415504	
Sin. 33° .. 2′ .. 25″½ - - - - - -	Log.	9·7365801	
Co. Sin. - - - - − 1″ - -	Log. ar. co.	− 10· ———	
Co. Sin. - - 25′ .. 47″ - -	Log. ar. co.	− 10·0000122	
- - - - - - - - - - Divide by 2 - - -		19·4781427	log. sum.
Sin. 33° .. 15′ .. 18″·19 - - - -	*Log.*	9·7390713	half sum.
Multiply by 2			

Product 66° .. 30′ .. 36″·38 ∠ required, agreeing to the $\frac{1}{100}$ part of a second with the angle *before found.*

Note.—This example is a partial statement of the angular distance, altitude, and depression of *Fiennes* and *Watten,* as observed at Calais in the Trigonometrical Survey of 1787, made for the purpose of connecting the observations of Greenwich and Paris: and here we cannot help expressing our unqualified opinion that the answer given with the question was a mere guess.

(Page 147.)

Ex. 2. Because the two measured distances D and d are equal, putting x for the height of the tower, and taking the formula given in Cor. to Prob. 4, page 129, vol. iii. it will be, $x =$

$$\sqrt{\frac{2d^2}{\cot^2.\text{A} + \cot^2.\text{C} - 2\cot^2.\text{B}}} = \frac{d}{\sqrt{\frac{1}{2}\cot^2.\text{A} + \frac{1}{2}\cot^2.\text{C} - \cot^2.\text{B}}}$$

$$= \frac{84}{\sqrt{\frac{1}{2}\cot^2.36^\circ..50' + \frac{1}{2}\cot^2.14^\circ - \cot^2.21^\circ..24'}}$$

Therefore by the rule page 129 as above,

---

* *If the log. sines to Radius* 1 *be used, the operation will be—*

Log. sin. 33° .. 28′ .. 13″½ is	− 1·7415504
Log. sin. 33° .. 2′ .. 25″½ is	− 1·7365801
Log. ar. co. cos. − 1″ + Log. ar. co. cos. 25′ .. 47″	0·0000122
Divide by 2	− 1·4781427 Log. sum
Sin. 33° .. 15′ .. 18·19 *Log.*	− 1·7390713 *as before.*

(Key to Vol. III. page 147.)

2 log. cot. 36°..50′=0·2510324 of which the nat. num. is 1·78251
2 log. cot. 14°.. 0′=1·2064578 - - - - - - - - - - - - 16·08635

Sum of the nat. numbers 17·86886

Half sum 8·93443
2 log. cot. 21°..24′=0·8136590 of which the nat. num. is 6·51117

And ·3844000 = log. of the remainder 2·42326

Subtract ·1922000 = half log. of the remainder.
From 1·9242793 log. of 84 feet=D.

log. of x = 1·7320793 *log. difference*, of which the natural number is 53·9609 feet. Ans.

INVESTIGATION OF GENERAL ROY'S RULE.

Ex. 3. Because in any spherical triangle the excess of its three angles above two right angles, is the measure of the area of the triangle, (*Cor.* 1. *Theor.* v. *Spher. Trig. vol.* iii. *p.* 84), and because the three angles of a spherical triangle − 180°, are to 180°, as the surface or area of the triangle, to the area of a great circle of the sphere*; if A, B, C, be severally put for the three angles, a for the area of the triangle, and q for the area of a great circle of the sphere, ($=\frac{1}{4}$ the surface of the sphere,) it will be, $\frac{a \times 180^\circ}{q} = \text{A}+\text{B}+\text{C}-180^\circ$; therefore, since the square of the radius multiplying 3·141592653589 79323846264338327950288419716939937510582097494459230781740629620899862803482534211706798214808651 32723066470938446319 &c.=the area of a great circle of the sphere=$\frac{1}{4}$ the superficies of the sphere, it follows that, $\frac{a \times 180^\circ}{\text{rad.}^2 \times 3{\cdot}141592\ \&\text{c.}} = \text{A}+\text{B}+\text{C}-180^\circ$. But the length of the earth's radius in feet, *(the earth considered a perfect sphere, and the length of a degree on its surface taken at* 365154·6 *feet,)* $= \frac{180^\circ \times 365154{\cdot}6}{3.1415926}$ feet; whence A+B+C

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* Theorem v. page 83, vol. iii.

(Key to Vol. III. page 147.)

$$-180^\circ=\frac{a\times180^\circ}{\overline{365154{\cdot}6}^2\times\overline{\frac{180^\circ}{3{\cdot}1415}}^2}\times3{\cdot}14159=\frac{3{\cdot}14159\times a}{180^\circ\times\overline{365154{\cdot}6}^2}$$

feet=the excess of A + B + C above two right angles. This excess expressed in seconds is $\frac{3{\cdot}14159\times60\times60\times a}{180^\circ\times\overline{365154{\cdot}6}^2}=\frac{62{\cdot}831853\times a}{\overline{365154{\cdot}6}^2}$. Now in logarithms the excess will be, log. $a+$ log. $62{\cdot}831853-2$ log. $365154{\cdot}6=$log. $a+1{\cdot}7981799-11{\cdot}1249536=$log. $a-9{\cdot}3267737$. *Which was required.*

Ex. 4. Half the sum of the sides being 22·5, and the three remainders (*see page* 29, *vol.* ii.) being 12·5, 5·5, and 4·5, half the sum of the logarithms of these 4 numbers is the log. of the area of the triangle.

But the Log. of 22·5. is 1·3521825
- - - - - - - - - 12·5. is 1·0969100
- - - - - - - - - 5·5. is 0·7403627
- - - - - - - - - 4·5. is 0·6532125

Log. sum = 3·8426677

Log. half sum = 1·9213338 which is the Log. of 83·432233 square miles.

Now $\overline{5280}^2$ feet = 1 square mile,
and 2 log. of 5280 feet=7·4452678

Log. sum 9·3666016 = the log. of the area of the triangle in feet.
Subtract the const. log. 9·3267737

The log. of the excess of the 3 angles above 180° is } ·0398279=L. of 1″·096.

mls. mls.
Ex. 5. 15 × 12=180 sq. mls. of which the log. is - - } - - - 2·2552725

And $\overline{1760\text{yds.}\times3}^2=\overline{5280}^2$ feet of which the log. is } - - - 7·4452678

Log. sum 9·7005403 the log. of the area of the triangle in feet.
Subtract the const. log. 9·3267737

Difference ·3737666 log. of 2″·3646
Wherefore 2″ .. 21‴ .. 52⁗½. Ans.

(Key to Vol. III. page 147.)

Otherwise.

24 miles in ft. = 126720, of which the log. is 5·1028452
And 15 miles in ft. = 79200, of which the log. is 4·8987252

Sum of the logs. 10·0015704
Deduct (see Art. 4, page 135, vol. iii.) 9·6278037

The com. log. of the spherical excess in seconds is 0·3737667 the log. of 2″..21‴..52′″½ *exactly as before.*

Ex. 6.

As the sum of the sides (41) - - - - - Log. ar. co. – 2·3872161
: difference of the sides (5) - - - - - - - - Log. 0·6989700
: : Cot. 29° .. 12′ .. 18″, (½ cont^{d} ∠) - - - - - Log. 10·2527696

: *Tang.* 12° .. 18′ .. 42″½ (*half diff. of the other* 2 $\angle^{les}$) 9·3389557

And the complement of half the included ∠ is 60°..47′..42″.

Hence the three angles of the plane triangle are 58°..24′..36″, 93° .. 6′ .. 24″½, and 48° .. 28′ .. 59″½. And the third side = 26·172355 miles.

Now half the sum of the sides = 33·586 miles; and the three remainders are 15·586, 10·586, and 7·403 (*page* 29, *vol.* ii.) But the

Log. of 33·586 is 1·5261583
- - - - 15·586 is 1·1927347
- - - - 10·586 is 1·0247319
- - - - 7·403 is 0·8694077

Log. sum 4·6130326

Half sum 2·3065163 log. of the area in miles.
And twice log. of 5280 is 7·4452678

Log. sum 9·7517841 log. of the area in feet.
Subtract the const. log. 9·3267737

Difference 0·4250104 the log. of 2″·6607.*

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* The same result will obtain, if the two given sides be converted into minutes and seconds, and the triangle solved spherically. (See Theor. 5, ch. iv.)

(Key to Vol. III. page 147.)

Ex. 7. As rad. $+\overset{\text{ft.}}{38}$: rad. : : given base : base required.

But the radius of the earth, at a mean calculation, is more than 21 millions of feet, therefore, as in *Prob.* ix. *page* 136, *vol.* iii. since the required base *(by multiplying means and extremes)* $= \frac{\text{rad.} \times \text{given base}}{\text{rad.} + 38\text{ft.}}$, or, *(by further comparison, and the division of the given base* × 38 *ft., by the rad.* + 38 *ft., expressed symbolically,)* $\text{B} - b = \text{B} \times (\frac{h}{r} - \frac{h^2}{r^2} + \frac{h^3}{r^3} - \&c.)$ Now neglecting the second and all the succeeding terms of the series as insignificant, it will be $\text{B} - b = \frac{\text{B}h}{r}$.

By Logarithms.

The log. of 34286 ft. is - - 4·5351168
And the log. of 38 ft. - - - 1·5797836

Log. sum 6·1149004
Deduct (l. rad. of the earth in ft.) 7·3223947

Remainder −2·7925057 the log. of ·062016267

Consequently 34286 − ·062016267 = 34285·937983732 ft. Ans.

Ex. 8. If the altitude be corrected by subtracting 44″ the refraction in altitude, and then by adding 6″ for the parallax, the true altitude of the sun is 52° .. 34′ .. 22″, the co. altitude 37°..25′..38″, the co. lat. 41°..9′, the co. declination 71°..30′, and the proportional hour angle 27° .. 8′ .. 30″. Wherefore,

As Sin. co. alt. 37°..25′..38″ Log. ar. co. − 10·2162728
: Sin. hour ∠ 27°.. 8′..30″ - - - - Log. 9·6591478
: : Sin. co.dec. 71 ..30′..00″ - - - - Log. 9·9769566

: *Sin.* ∠ *reqd. or of its supplement* - Log. 9·8523772 the log. sin. of 45° .. 23′ .. $2''\frac{5}{6}$. Ans.

Ex. 9. The part of the ecliptic between the beginning of *Aries* and the center of the sun being considered as the hypothenuse of a right angled spherical triangle, and the obliquity of the ecliptic as the angle adjacent to the side sought; the Tang. of the sun's right ascension = Tang of his longitude $\times^{d}$ by the cosine of the obliquity of the ecliptic. But

(Key to Vol. III. page 147.)

Log. tang. 29°..13′..43″ is 9·7478285
Log. cosine 23°..27′..40″ is 9·9625259

Their sum (abating 10 in the index) is 9·7103544 which is the log. tang. of 27°.. 10′.. 13″.. 50″$\frac{1}{2}$ = the sun's right ascension. Q. E. I.

Ex. 10. By Theorem 11 in the Scholium to Prob. xii. alluded to in the question, the cosine of the sun's longitude=co. sin. of his right ascension $\times^{d}$ by the cosine of his declination. Now the

Log. co. sin. 32°.. 46′.. 52″$\frac{1}{2}$ is 9·9246636
Log. co. sin. 13°.. 13′.. 27″ is 9·9883281

Their sum (abating 10 in the index) is 9·9129917 which is the log. cos. of 35°.. 4′.. 12″.. 31‴ = the sun's longitude. Q. E. I.

Ex. 11. The Theorem for the longitude being (*see Theorem* iii. *page* 142, *vol.* iii.) "that its tangent = the sine of the obliquity of the ecliptic $\times^{d}$ by the tangent of the declination $\times^{d}$ by the secant of the right ascension + the tangent of the right ascension × cosin. of the obliquity of the ecliptic;" it will be by logarithms as follows, *to radius* 1 :

Log. sin. 23°..27′..40″ is –1·6000212
Log. tang. 62°..50′.. 0″ 0·2897176
Log. sec. 162°..50′..34″ 0·0197698

Log. sum –1·9095086 the com. log. of ·8119113

And,

Log. tang. 162°..50′..34″ is –1·4895840
Log. co. sin. 23°..27′..40″ –1·9625259

Log. sum –1·4521099 the com. log. of ·2832108

The sum of the two natural numbers is 1·0951221

Which is the natural tangent of 132°.. 24′.. 2″=*longitude of α Ursæ Majoris.*

Again. The Theorem for the latitude (*see Theorem* iv. *page* 142, *vol.* iii.) being "that its sine=sin. of the declination × co. sin. of the obliquity of the ecliptic, – sin. of the right ascension × co. sin. of the declination × sin. of the eliptic's inclination to the equator," the solution will be by logarithms as follows:

(Key to Vol. III. page 147.)

Log. sin. 62°..50′ is – 1·9492349
Log. co. sin. 23°..27′..40″ is – 1·9625259

Log. sum – 1·9117608 the com. log. of ·8161328

And,

Log. sin. 162°..50′..34″ is – 1·4698142
Log. co. sin. 62°..50′.. 0″ is – 1·6595173
Log. sin. 23°..27′..40″ is – 1·6000212

Log. sum – 2·7293527 the com. log. of ·0536232

The *difference* of the two natural numbers is ·7625096

Which is the nat. sin. of 49° .. 41′ .. 9″ = *lat. of α Ursæ Majoris.* Q. E. I.

Ex. 12. Putting m for 60833, M for 60494, also l for 49° .. 3′ and L for 12° .. 32′, the Theorem (page 114) where d=the *greater*, and c the *less* axis of the earth, is

$$\frac{d}{c}=\sqrt{\frac{(\mathrm{M}^{\frac{1}{3}}\,\text{sin. }\mathrm{L}+m^{\frac{1}{3}}\,\text{sin. }l).(\mathrm{M}^{\frac{1}{3}}\,\text{sin. }\mathrm{L}-m^{\frac{1}{3}}\,\text{sin. }l)}{(m^{\frac{1}{3}}\,\text{co. sin. }l+\mathrm{M}^{\frac{1}{3}}\,\text{co. sin. }\mathrm{L}).(m^{\frac{1}{3}}\,\text{co. sin. }l-\mathrm{M}^{\frac{1}{3}}\,\text{co. sin. }\mathrm{L})}}$$

which may be more simply expressed, As $d : c :: $ sq. root of the rectangle of the sum and difference of $\mathrm{M}^{\frac{1}{3}}$ sin. L and $m^{\frac{1}{3}}$ sin. l. : sq. root of the rectangle of the sum and difference of $\mathrm{M}^{\frac{1}{3}}$ co. sin. L and $m^{\frac{1}{3}}$ co. sin. l. Now

Log. $\mathrm{M}^{\frac{1}{3}}$ is 1·5939041
Log. sin. L is – 1·3364794

Log. sum ·9303835 com. log. of 8·5189.

Log. $m^{\frac{1}{3}}$ is 1·5947131
Log. sin. l – 1·8781090

Log. sum 1·4728221 com. log. of 29·7046.

The *sum* of these two natural numbers is 38·2235
Their *difference* is 21·1857

The log. of 38·2235 is 1·5823300
The log. of 21·1857 is 1·3260285

Log. sum 2·9083585

Half sum 1·4541792 com. log. of 28·456.

Again,

Log. $m^{\frac{1}{3}}$ is 1·5947131
Log. co. sin. *l.* is −1·8165066

Log. sum 1·4112197 c. log. of 25·776.

Log. $\text{M}^{\frac{1}{3}}$ is 1·5939041
Log. co. sin. L −1·9895254

Log. sum 1·5834295 c. log. of 38·320.

The *sum* of these two natural numbers is 64·096
Their *difference* is 12·544
The log. of 64·096 is 1·8068309
The log. of 12·544 is 1·0984360

The half sum of these logs. is 1·4526334 com. log. of 28·355
Therefore, As $d : c :: 28456 : 28355$ ∵ ratio of the axes. Q. E. I.

Ex. 13. Let R be a degree of the earth's equator, M and *m* respectively the first and last degrees of latitude; then R, M, *m*, are severally to each other as the radii of their osculatory circles; in the same ratio are AC, $\frac{\text{CD}^2}{\text{AC}}$, and $\frac{\text{AC}^2}{\text{CD}}$. *See the fig. to Prob.* xiii. *page* 143, *vol.* iii.

Hence, As $\text{R}^2 : m^2 :: \text{AC}^2 : \frac{\text{AC}^4}{\text{CD}^2}$. And,

As $\text{R} : \text{M} :: \text{AC} : \frac{\text{CD}^2}{\text{AC}}$. By composition, therefore, $\text{R}^3 : \text{M}m^2 :: 1 : 1$. Wherefore

$\text{R} = \sqrt[3]{\text{M}m^2}$, that is, R is the first of two mean proportionals between M and *m*. *Which was to be proved.*

Ex. 14. The length of a degree of the meridian in any latitude being to the length of a degree of the meridian in any other latitude, in the direct ratio of the radii of the respective osculatory circles; if R be put for the radius of curvature at any point of the terrestrial meridian, *c*, *d*, and *e* being as in Proposition xiii. Trigonometrical Surveying, vol. iii. of the COURSE, it will be

(Key to Vol. III. page 147.)

$$\text{R} = \frac{d^2}{c\,(d^2 - e^2 \sin^2.\,\text{L})^{\frac{3}{2}}} \because \left(1 - \frac{e^2}{d^2} \times \sin^2.\,\text{L}\right)^{-\frac{3}{2}} \because 1 + \frac{\frac{3}{2}e^2}{d^2} \times \sin^2.\,\text{L} + \frac{\frac{15}{4}e^4}{d^4} \times \sin^4.\,\text{L} + \&c. \because 1 + \frac{\frac{3}{2}e^2}{d^2} \times \sin^2.\,\text{L} \textit{ very nearly}.$$

Therefore the increase of a degree of the terrestrial meridian in receding from the equator, is in the ratio of $\frac{3e^2}{2d^2} \times \sin^2.\,\text{L}$ *nearly,* that is, very nearly in the duplicate ratio of the sine of the latitude. Q. E. D.

(Page 148.)

Ex. 15.* Because p and m are given, the radii of curvature to p and m are determinable. Let these be r and $r+n$; the radius of curvature to d will be $\frac{r\,(r+n)}{(r+n) - n.\,\sin^2.\,a}$. Now to compare the degrees of the three several arcs whereof the radii of curvature are r, $r+n$, and $\frac{r\,(r+n)}{(r+n) - n.\,\sin^2.\,a}$, substitute the given lengths d, m, and p $(=m+v)$ for the radii, (since the degrees are in the direct ratio of the radii of curvature,) and it will be, $d = \frac{m\,(m+v)}{(m+v) - v.\,\sin^2.\,a} = \frac{m \times p}{p - v.\,\sin^2.\,a} = \frac{pm}{p + (m-p)\,\sin^2.\,a}$. Q. E. D.

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* At page 45, No. 98 of the Ladies' Diary, an example proposed in 1800 by Capt. (now Col.) Mudge, and applicable to this Theorem, may be seen solved, and the Theorem investigated at more length.

POLYGONOMETRY.

(Page 160.)

Ex. 6. Because twice the area of the hexagon $= \Big\{$ AB. BC. sin. B. + AB. DC. sin. (B + C) + BC. DC. sin. C + DE. EF. sin. E + DE.

(Key to Vol. III. page 160.)

$\left.\text{AF}.\,\sin.\,(\text{E}+\text{F})+\text{EF}.\ \text{AF}.\sin.\,\text{F}.\right\} = \left\{1284\times1782\times\sin.32^\circ+1284\times2400\times\sin.80^\circ+1782\times2400\times\sin.48^\circ+2700\times2860\times\sin.66^\circ+2700\times4621{\cdot}5\times\sin.\,25^\circ..28'..22''+2860\times4621{\cdot}5\times\sin.\,88^\circ..31'..38''.\right\}$ it will be by logarithms,

1284	log.	3·1085650		
1782	log.	3·2509077		
32°	log. sin.	−1·7242097		
	Log. sum	6·0836824	the log. of	1212501·95

1284	log.	3·1085650		
2400	log.	3·3802112		
80°	log. sin.	−1·9933515		
	Log. sum	6·4821277	the log. of	3034882·63

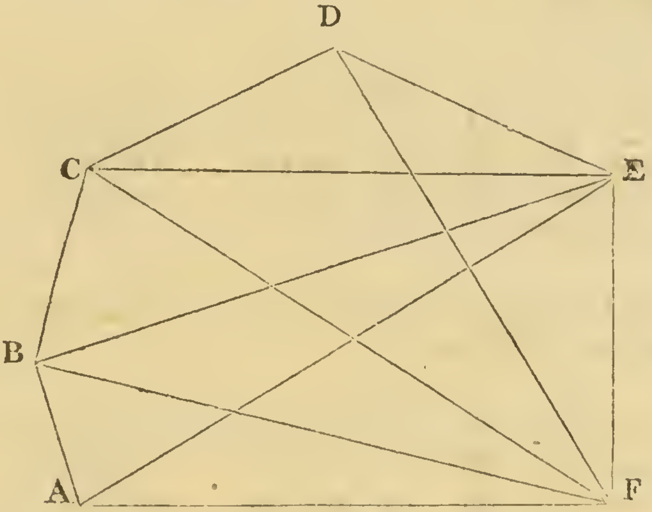
1782	log.	3·2509077		
2400	log.	3·3802112		
48°	log. sin.	−1·8710735		
	Log. sum	6·5021924	the log. of	3178281·75

2700	log.	3·4313638		
2860	log.	3·4563660		
66°	log. sin.	−1·9607302		
	Log. sum	6·8484600	the log. of	7054400·

2700	log.	3·4313638		
4621·5	log.	3·6647830		
25° .. 28′ .. 22″	log. sin.	−1·6335514		
	Log. sum	6·7296982	the log. of	5366587·65

2860 log. 3·4563660
4621·5 log. 3·6647830
88° .. 31′ .. 38″ log. sin. −1·9998565
Log. sum 7·1210055 the log. of 13213124·61
Sum of the nat. numbers 33059778·59
Half sum and AREA required 16529889·29

Which problem solved by the formula, for the surface or area' in page 155, third vol. of the Course, will be as follows:



Let ABCDEF be the hexagon given in Ex. 1. Then by plane trigonometry,

∠EDF = 34° .. 4′ .. 14″ BF = 5136·3
∠FAE = 31° .. 20′ .. 23″ CE = 4585·7
∠ABF = 59° .. 36′ .. 24″
∠ECD = 27° .. 38′ .. 36″
∠CED = 24° .. 21′ .. 24″
∠AFB = 13° .. 51′ .. 58″
∠BCF = 72° .. 18′ .. 20″
∠BFC = 19° .. 18′ .. 4″
∠CFD = 26° .. 22′ .. 34″
∠FCE = 32° .. 3′ .. 4″
∠EBC = 56° .. 41′ .. 31″
∠BEC = 18° .. 57′ .. 5″
∠AFE = 91° .. 28′ .. 22″ = a'
∠BFE = 77° .. 36′ .. 24″ = b'
∠CFE = 58° .. 18′ .. 20″ = c'
∠DFE = 31° .. 55′ .. 46″ = d'
∠AEF = 57° .. 11′ .. 15″ = a''
∠BEF = 70° .. 41′ .. 31″ = b''
∠CEF = 89° .. 38′ .. 36″ = c''
∠DEF = 114° .. ——— = d''

(Key to Vol. III. page 160.)

Wherefore, by Prob. 1, Polygonometry, the area ABCDEF =

$$\tfrac{1}{2}\mathrm{FE}^2 \times \left\{ \frac{\text{Sin.}\,a''.\ \text{Sin.}\,b''.\ \text{Sin.}\,(a'-b')}{\text{Sin.}\,(a'+a'').\ \text{Sin.}\,(b'+b'')} + \frac{\text{Sin.}\,b''.\ \text{Sin.}\,c''.\ \text{Sin.}\,(b'-c')}{\text{Sin.}\,(b'+b'').\ \text{Sin.}\,(c'+c')} + \frac{\text{Sin.}\,c''.\ \text{Sin.}\,d''.\ \text{Sin.}\,(c'-d')}{\text{Sin.}\,(c'+c'').\ \text{Sin.}\,(d'+d'')} + \frac{\text{Sin.}\,d'.\ \text{Sin.}\,d''}{\text{Sin.}\,(d'+d'')}. \right\}$$

which admits of the following solution by logarithms.

$a'' = 57° .. 11' .. 15''$ log. sin. 9·9245111
$b'' = 70° .. 41' .. 31''$ log. sin. 9·9748590
$(a'-b') = 13° .. 51' .. 58''$ log. sin. 9·3795850
$(a'+a'') = 148° .. 39' .. 37''$ log. sin. – 10·2839042 ar. co.
$(b'+b'') = 148° .. 17' .. 55''$ log. sin. – 10·2794337 ar. co.

Log. sum rejecting 10 *in the index* – 1·8422930 which is the logarithm of ·69549.

$b'' = 70° .. 41' .. 31''$ log. sin. 9·9748590
$c'' = 89° .. 38' .. 36''$ log. sin. 9·9999916
$(b'-c') = 19° .. 18' .. 4''$ log. sin. 9·5192144
$(b'+b'') = 148° .. 17' .. 55''$ log. sin. – 10·2794337 ar. co.
$(c'+c'') = 147° .. 56' .. 56''$ log. sin. – 10·2951710 ar. co.

Log. sum rejecting 10 *in the index* 0·0486697 which is the logarithm of 1·1191.

$c'' = 89° .. 38' .. 36''$ log. sin. 9·9999916
$d'' = 114°$ log. sin. 9·9607302
$(c'-d') = 26° .. 22' .. 34''$ log. sin. 9·6476388
$(c'+c'') = 147° .. 56' .. 56''$ log. sin. – 10·2951710 ar. co.
$(d'+d'') = 145° .. 55' .. 46''$ log. sin. – 10·2516465 ar. co.

Log. sum rejecting 10 *in the index* 0·1351781 which is the logarithm of 1·3651.

$d' = 31° .. 55' .. 46''$ log. sin. 9·7233527
$d'' = 114°$ log. sin. 9·9607302
$(d'+d'') = 145° .. 55' .. 46''$ log. sin. – 10·2516465 ar. co.

Log. sum rejecting 10 *in the index* – 1·9357294 which is the logarithm of ·86244.

The sum of these 4 natural numbers is 4·04213, of
which the log. is 0·6066103
$\frac{1}{2}\text{FE}^2=4089800$; its log. is 6·6117021

Log. sum 7·2183124 which is the logarithm of 16531505·7, the AREA REQUIRED.

☞ We could have solved this question in various other ways, and with much less trouble, but we thought it proper to find the area by *Lhuillier*'s theorem as well as by reducing the expression for the surface given in Problem I. Although the results do not exactly agree, (which indeed may be owing to some trifling error which we are not at present aware of,) yet the difference is not comparatively great.

Ex. 7. $$\text{Tang.}\ \angle\text{D} = \frac{\text{BC. sin. C} + \text{AB. sin. (B} + \text{C)}}{\text{DC} + \text{CB. cos. C} + \text{AB. cos. (B} + \text{C)}} =$$
$$\frac{30 \times \sin. 82° \mathinner{..} 37' + 24 \times \sin. 170° \mathinner{..} 19'}{34 + 30 \times \cos. 82° \mathinner{..} 37' + 24 \times \cos. 170° \mathinner{..} 19'} =$$
$$\frac{30 \times \sin. 82° \mathinner{..} 37' + 24 \times \sin. 9° \mathinner{..} 41'}{34 + 30 \times \cos. 82° \mathinner{..} 37' - 24 \times \cos. 9° \mathinner{..} 41'}.$$ Now the

Log. of 30 is 1·4771213
Log. sin. of 82°..37′ is −1·9963841

Sum 1·4735054 log. of 29·751

Log. of 24 is 1·3802112
Log. sin. of 9°..41′ is −1·2258328

Sum 0·6060440 log. of 4·0369

33·7879 Numerator.

Log. of 30 is 1·4771213
Log. cos. of 82°..37′ is −1·1089272

Sum 0·5860485 log. of 3·8552

Log. of 24 is 1·3802112
Log. cos. of 9°..41′ is −1·9937679

Sum 1·3739791 log. of 23·658

But 34 + 3·8552 − 23·658 = 14·1972 Denominator.

Again, the log. of 33·7879 is 1·5287611
log. of 14·1972 is 1·1512027

Log. difference 0·3775584
Add 10 to the index 10

Sum 10·3775584 the log. tangent of 67° .. 14′ = ∠ADC. Wherefore [*Eucl. B.* I. *Prop.* xxxii. *Cor.* i.] ∠DAB = 103° .. 5′.

Now the side AD = AB. cos. 103° .. 5′ + BC. cos. 9° .. 41′ + CD. cos. 67° .. 14′ = −24 × cos. 76° .. 55′ + 30 × cos. 9° .. 41′ + 34 × cos. 67° .. 14′.

In Logarithms.

The log. of −24 is 1·3802112
Log. cos. 76° .. 55′ is −1·3548150

Log. sum ·7350262 the log. of −5·4328

The log. of 30 is 1·4771213
Log. cos. 9° .. 41′ is −1·9937679

Log. sum ·4708892 the log. of 29·572

The log. of 34 is 1·5314789
Log. cos. 67° .. 14′ is −1·5876876

Log. sum 1·1191665 the log. of 13·157

Sum = AD = 37·2962

Now conceiving a line to be drawn from A to C, dividing the trapezium into two triangles, the double area will be

AB. BC. sin. B. + CD. DA. sin. D. = 24 × 30 × sin. 87° .. 42′ + 34 × 37·2962 × sin. 112° .. 46′.

In Logarithms.

The log. of 24 is 1·3802112
log. of 30 is 1·4771213
Log. sin. 87° .. 42′ is −1·9996500

Log. sum 2·8569825 the log. of 719.42

(Key to Vol. III. page 160.)

The log. of 34 is 1·5314789
log. of 37·2962 is 1·5716646
Log. sin. 112° .. 46′ is −1·9647726

Log. sum 3·0679161 the log. of 1169·27

Sum of the nat. numbers 1888·69

Half sum and Area 944·34

Otherwise without a diagonal.

The double area is AB. BC. sin. B. + AB. DC. sin. (B + C) + BC. DC. sin. C = 24 × 30 × sin. 87° .. 42′ + 24 × 34 × sin. 170° .. 19′ + 30 × 34 × sin. 82° .. 37′.

In Logarithms.

The log. of 24 is 1·3802112
log. of 30 is 1·4771213
Log. sin. 87° .. 42′ is −1·9996500

Log. sum 2·8569825 log. of 719·42

The log. of 24 is 1·3802112
log. of 34 is 1·5314789
Log. sin. 170°..19′ is −1·2258328

Log. sum 2·1375229 log. of 137·253

The log. of 30 is 1·4771213
log. of 34 is 1·5314789
Log. sin. 82° .. 37′ is −1·9963841

Log. sum 3·0049843 log. of 1011·54

Sum of the nat. numbers 1868·213

Half sum and Area 934·106

Or, by drawing a diagonal from B to D.

Twice the area will be BC. CD. sin. C + DA. AB. sin. A = (30 × 34) × sin. 82° .. 37′ + (37·2962 × 24) × sin. 76° .. 55′.

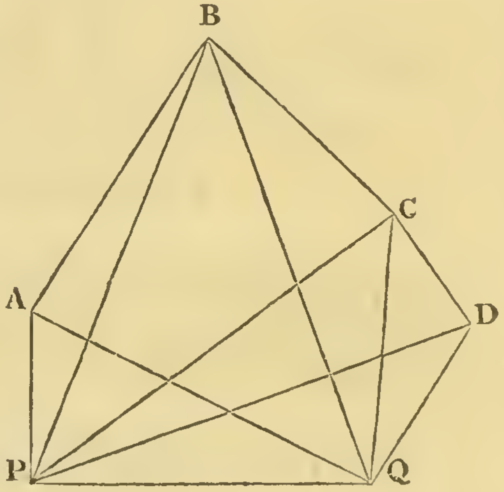
(Key to Vol. III. page 160.)

		In Logarithms.	
The log. of 30	is	1·4771213	
log. of 34	is	1·5314789	
Log. sin. 82°..37′	is	–1·9963841	
Log. sum		3·0049843	log. of 1011·5

The log. of 37·2962	is	1·5716646	
log. of 24	is	1·3802112	
Log. sin. 76°..55′	is	–1·9885776	
Log. sum		2·9404534	log. of 871·87
		Sum of the nat. numbers	1883·37
		Half sum and AREA	941·68*

* *The result by these methods is by no means coincident, and proves the necessity of ascertaining the sides and angles more minutely accurate than in common cases is usual.*

Ex. 8. Here $a' = 89^\circ..14'$
$b' = 68^\circ..11'$
$c' = 36^\circ..24'$
$d' = 19^\circ..57'$
$a'' = 25^\circ..18'$
$b'' = 69^\circ..24'$
$c'' = 94^\circ..\ 6'$
$d'' = 121^\circ..18'$
$(a' - b') = 21^\circ..\ 3'$
$(a' + a'') = 114^\circ..32'$
$(b' + b'') = 137^\circ..35'$
$(b' - c') = 31^\circ..47'$
$(c' + c'') = 130^\circ..30'$
$(c' - d') = 16^\circ..27'$
$(d' + d'') = 141^\circ..15'$



Wherefore it is,

+ log. sin. 25°..18′	- - - - -	1·6307917
+ log. sin. 69°..24′	- - - - -	1·9713035
+ log. sin. 21°.. 3′	- - - - -	1·5549868
– log. sin. 114°..32′	ar. co.	0·0410923
– log. sin. 137°..35′	ar. co.	0·1710070
Log. sum		–1·3691813 log. of ·23398

(Key to Vol. III. page 160.)

+ log. sin. 69°..24′	- - - -	−1·9713035	
+ log. sin. 94°.. 6′	- - - -	−1·9988871	
+ log. sin. 31°..47′	- - - -	−1·7215704	
− log. sin. 137°..35′	ar. co.	0·1710070	
− log. sin. 130°..30′	ar. co.	0·1189545	
	Log. sum	−1·9817225	log. of ·95879

+ log. sin. 94°.. 6′	- - - -	−1·9988871	
+ log. sin. 121°..18′	- - - -	−1·9316911	
+ log. sin. 16°..27′	- - - -	−1·4520603	
− log. sin. 130°..30′	ar. co.	0·1189545	
− log. sin. 141°..15′	ar. co.	0·2034788	
	Log. sum	−1·7050718	log. of ·50707

+ log. sin. 19°..57′	- - - -	−1·5330090	
+ log. sin. 121°..18′	- - - -	−1·9316911	
− log. sin. 141°..15′	ar. co.	0·2034788	
	Log. sum	−1·6681789	log. of ·46578
		Sum of the nat. numbers	2·16562

Now $\frac{1}{2}$ PQ^2=3220722, its log. 6·5079533
The logarithm of 2·16562 is 0·3355822

Log. sum 6·8435355 which is the logarithm of 6974859·677 the area in square links, which divided by 100000, the number of square links in an acre, the quotient is 69·74859677 acres. Ans.

DIVISION OF SURFACES.

(Page 172.)

Ex. 1. $\frac{20+18+16}{2} = 27$, and

(Key to Vol. III. page 172.)

$$27\begin{Bmatrix}-20\\-18\\-16\end{Bmatrix}=\begin{Bmatrix}7\\9\\11\end{Bmatrix}\text{the three remainders.}$$

Also $\sqrt{27\times7\times9\times11}=136\cdot8$ *nearly*. Therefore the proposed triangular field contains 13·68 acres *nearly*.

Consequently 9·68 ac. and 4 ac. are the two intended divisions.

Let the side 20, be AB; 18, BC, and 16, AC. Likewise let the 4 acres be contiguous to AC; and let D be the termination of the fence in the side AB. Then

Acres Acres Chains Chains
As 13·68 : 4 : : 20 : 5·848=AD } the two distances of the dividing line.
Hence 14·152=BD }

Again,

The angle $A=58°..45'$ because $\frac{400+256-324}{2\times16\times20}=\cdot51875$ = nat. co. sin. 58° .. 45′. And

As 21·848 (AC+AD) - - - - - - - - - - Log. ar. co. −2·6605883
: 11·848 (AC−AD) - - - - - - - - - - - - - Log. 1·0736450
: : Cot. 29° .. 22′½ (*half the contained* ∠*le*) - - Log. 10·2495716

: *Tang.* 43°..56′ (*half the diff. of the* ∠*les* ADC, ACD) *Log.* 9·9838049

Whence the angle ADC=104° .. 33′½. But

As Sin. 104° .. 33′½ - - - Log. ar. co. −10·0141730
: Sin. 58° .. 45′ - - - - - - - - - Log. 9·9319213
: : 16 chains - - - - - - - - - - - - Log. 1·2041200

: 14·132 chains = CD - - - - - - - *Log.* 1·1502143

Therefore the length of the dividing line is 14·132 chains, and its distance from the angle which is neither most nor least acute, 5·848 chains.

Ex. 2. $\frac{5+12+13}{2}=15$. And

$$15-\begin{Bmatrix}5=10\\12=3\\13=2\end{Bmatrix}\text{the three remainders of the half sum and sides.}$$

Also $\sqrt{15\times10\times3\times2}=30$ the area of the triangle.

Therefore $m=20$, and $n=10$ adjacent to the longest side, AB.

Now let the side 12 be BC, and F the point in it to be determined. $\text{CF}=\text{CB}\sqrt{\frac{m}{m+n}}=12\sqrt{\frac{20}{30}}=12\sqrt{\frac{2}{3}}=4\sqrt{6}=$ $9\cdot7979588$.

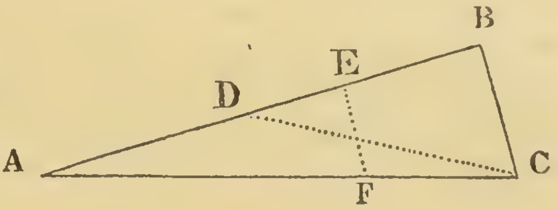
Again. Let the other extremity of the dividing line be G, and it will be, As CF : FG : : CB : AB. Hence

$$\frac{9\cdot7979588\times13}{12}=10\cdot6144553\dot{6}=\text{FG}.$$

Wherefore the length of the dividing line is $10\cdot6144553\dot{6}$, and the distance of its extremity in BC from the longest side is 2·2020412; and of its other extremity $\frac{5}{12}$ (2·2020412), or $\cdot9175171\dot{6}$, which were to be determined.*

* This and the last example admit, each, of two answers.

Ex. 3. Construct a triangle having its three sides respectively equal to the three given lines, namely, AB=24, AC=25, and BC=7. Bisect AB in D, and join CD, the triangle ABC is bisected by CD [*Eucl.* i. 38.].



Also $\sqrt{\text{AD}\times\text{AC}}=\sqrt{12\times25}=\sqrt{300}=17\cdot3205081=$ the distance of the required line (to be taken in AC and AB) from the point A the angular point of the acutest angle of the given triangle. [*Eucl.* i. 19.] If therefore in AC, AF be taken equal $\sqrt{300}$, and from AB there be cut a part AE equal to AF, the straight line joining the points E and F will be the shortest line possible that can divide the triangle ABC into two equal parts. (*For the demonstration, see page* 163, *vol.* iii. *of the* COURSE.)

To ascertain the length of this line, EF, the angle A = 16°..15′ because $\frac{25^2+24^2-7^2}{2(24\times25)}=\cdot96=$nat.co. sin. 16°..15′ *nearly*. Therefore the angle AEF=$\frac{1}{2}$ the supplement of 16°..15′, or 81°..52′$\frac{1}{2}$.

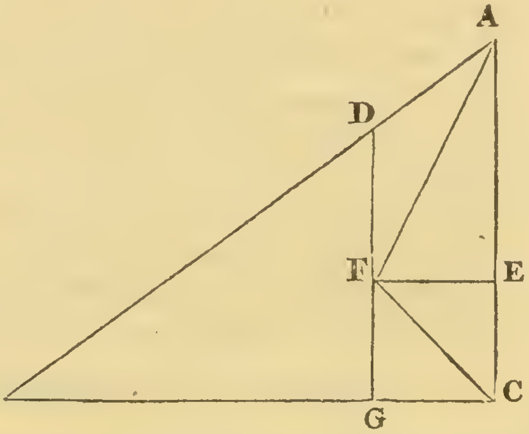
And

As Sin. 81°..52′$\frac{1}{2}$ - - - Log. ar. co. $-10\cdot0043815$
: Sin. 16°..15′ - - - - - - - - - Log. 9·4468927
: : 17·3205081=AF - - - - - - - Log. 1·2385607

: 4·89592=EF - - - - - - - - - *Log.* ·6898349

(Key to Vol. III. page 172.)

Consequently the position and length of the proposed shortest-possible line are determined. Q. E. F.



Ex. 4. Construct a triangle ABC right-angled at C, and having the side AC $=6$ and the side BC $=8$, the hypothenuse AB will necessarily be 10; which three numbers are the three sides given. But the area of the triangle $=\frac{8\times6}{2}=24=16(1\cdot5)$. Therefore $7\times1\cdot5$, and $9\times1\cdot5$ are the two divisions proposed. Let $m=$the former, $n=$the latter; and assume in BC any point G through which a line parallel to AC shall give the trapezium ADGC to the triangle BDG, as m is to n. BG$=$BC$\sqrt{\frac{n}{m+n}}$ [*Case* 2. *Prob.* 1. *chap.* vii. *vol.* iii.] $=8\sqrt{\frac{9}{16}}=\frac{24}{4}=6$. Consequently GC$=2$. Parallel, therefore, to AC, draw DG, so that GC may be equal to 2, the line DG is the line required. For,

As $10:8::\sin.90:\sin.53^\circ..7'..48''\frac{1}{3}=$ the angle BAC, of which the half is $26^\circ..33'..54''\frac{1}{6}$; and the half of the angle ACB is 45°.

Now let AF and CF be two lines bisecting the angles A and C, F is the center of the inscribed circle to the triangle ABC. But

As Sin. $108^\circ..26'..5''\frac{5}{6}$ ($=\angle$AFC) Log. ar. co. $-$ 10·0228788
: Sin. $26^\circ..33'..54''\frac{1}{6}$ ($=\angle$FAC) - - - - Log. 9·6505149
: : 6 ($=$side AC) - - - - - - - - - - - - - Log. 0·7781513

: $2\cdot8284271=\sqrt{8}=\sqrt{2^2+2^2}=$FC - - - *Log.* 0·4515450

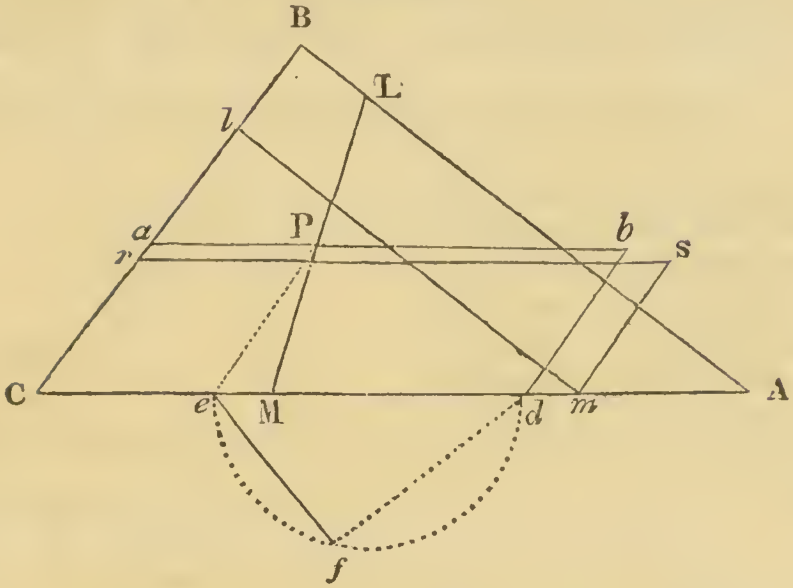
That is, the point F is in the line GD, *which was to be proved.*

Hence the length of the dividing line is $4\frac{1}{2}$.

It is evident that a line drawn through the point E *parallel to* BC *would not divide the triangle* ABC *in the ratio required, but the position of a line parallel to* BC *to effect this is determinable by Case* 2. *Prob.* I. *See* lm *in the following diagram.*

(Key to Vol. III. page 172.)

OTHERWISE.



Construct the given triangle right-angled at B and parallel to AB the side which is greater than BC draw *m l* dividing the given triangle in the ratio of *m* to *n* the triangle *m l* C being represented by *m* (=9) and the trapezium A B *lm* by *n* (=7.) [*Case* 2. *Theor.* 1. *Chap.* vii. *vol.* iii.].

Bisect *l* C in *r* and complete the rhomboid *rsm* C. Find the center P of the inscribed circle to ABC, [*Eucl.* iv. 4.] and draw P*e* parallel to BC. Make C*a* equal to C*e*, and take C*d* to C*m* as C*r* to C*a*; also complete the rhomboid C*abd*, which will be equal to the former rhomboid, and of which the side *ab* will pass through P. [*Eucl.* vi. 23.] Upon *de* as a diameter describe the semi-circle *dfe*, and apply in it *ef* equal to *e* C equal to C*a*. Join *df*, and make *d* M equal to *df*; lastly through the points P and M draw ML; ML divides the triangle ABC in the proposed ratio; and might also be proved to be the least possible line that can partition the given triangle in that ratio. (*For demonstration see page* 162, *vol.* iii.)

Ex. 5. As 31 (*sum of the given sides*) - - Log. ar. co. $-$2·5086383
: 3 (*difference of the given sides*) - - Log. ·4771213
: : Cot. 35° (*half the contained* $\angle^{le}$) - - Log. 10·1547732

: *Tang.* 7° .. 52′ .. 8″ - - - - - - - - - *Log.* 9·1405328

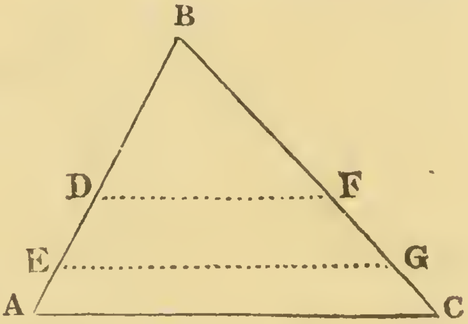
Consequently the two angles at the base of the triangle are, 47° .. 7′ .. 52″ and 62° .. 52′ .. 8″.

(Key to Vol. III. page 172.)

Again. As Sin. 62° .. 52′ .. 8″ - - Log. ar. co. −10·0506269
: Sin. 70° - - - - - - - - - - - - Log. 9·9729858
: : 17 (*side subt.* ∠ 62° .. 52′ .. 8″) Log. 1·2304489

: 17·95 *base and longest side of the* Δ. *Log.* 1·2540616

Constructing, then, a triangle of which the three sides are respectively equal to 14, 17, and 17·95, (which let be ABC,) it is required to divide it into three equal parts by lines parallel to AC. Assume D and E in AB, as the extremities of the lines sought.



$$\left.\begin{aligned}\text{BD}&=\text{BA}\sqrt{\tfrac{1}{3}}=14\sqrt{\tfrac{1}{3}}=\tfrac{14}{3}\sqrt{3}=8{\cdot}0829037\\ \text{BE}&=\text{BA}\sqrt{\tfrac{2}{3}}=14\sqrt{\tfrac{2}{3}}=\tfrac{14}{3}\sqrt{6}=11{\cdot}4309519\end{aligned}\right\}$$

Take, therefore, BD= 8·0829037, and BE=11·4309519, and parallel to AC draw DF and EG. It will be,

As BA : AC : : BE : EG, and
As BA : AC : : BD : DF, that is,

$$\left.\begin{aligned}&\text{As } 14 : 17{\cdot}95 :: 11{\cdot}4309519 : 14{\cdot}6561133=\text{EG}\\ &\text{As } 14 : 17{\cdot}95 :: 8{\cdot}0829037 : 10{\cdot}3634374=\text{DF}\end{aligned}\right\}\text{Ans.}$$

Ex. 6. Suppose 56° one of the angles of the triangle, then 66° .. 3′ is the 3d angle. Also suppose 58° to be one of the angles, in which case the 3d angle is 64° .. 3′. It will be,

As Sin. 57° .. 57′ - - Log. ar. co. −10·0718166
: Sin. 56° - - - - - - - - - - Log. 9·9185742
: : 112·65 - - - - - - - - - - Log. 2·0517312

: 110·19 - - - - - - - - - - *Log.* 2·0421220

And

As Sin. 57° .. 57′ - - Log. ar. co. −10·0718166
: Sin. 66° .. 3′ - - - - - - - Log. 9·9608987
: : 112·65 - - - - - - - - - - Log. 2·0517312

: 121·46 - - - - - - - - - - *Log.* 2·0844465

Now 121·46 − 110·19 = 11·27, but should be 8. Again,

(Key to Vol. III. page 172.)

As Sin. 57° .. 57′ - - Log. ar. co. −10·0718166
: Sin. 58° - - - - - - - - - - - Log. 9·9284205
: : 112·65 - - - - - - - - - - - Log. 2·0517312

: 112·71 - - - - - - - - - - - *Log.* 2·0519683

And

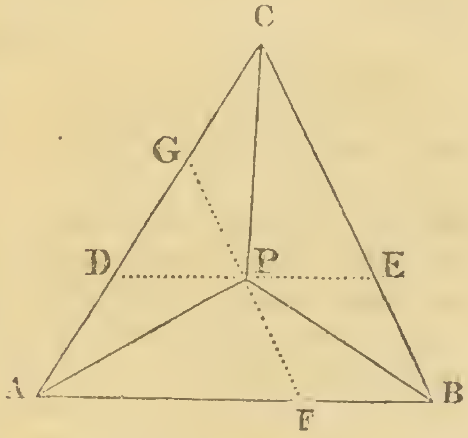
As Sin. 57° .. 57′ - - Log. ar. co. −10·0718166
: Sin. 64° .. 3′ - - - - - - - Log. 9·9538448
: : 112·65 - - - - - - - - - - - Log. 2·0517312

: 119·51 - - - - - - - - - - - *Log.* 2·0773926

And 119·51 − 112·71 = 6·8, but should be 8.

In the first instance the error is 3·27 *too much*, in the 2d it is 1·2 *too little*. The *sum* of the errors is 4·47. But,

As 4·47 : 2° : : 1·2 : 0° .. 32′·22 the *excess* of 58°, Therefore the three angles are 57° .. 57′, 57° .. 28′, and 64° .. 35′. Hence the three sides of the triangle are 112·65, 112·05, and 120·05, *nearly*.



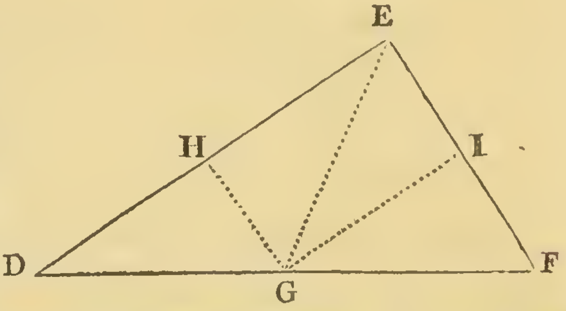
Construct, then, a triangle, as ABC, having its three sides equal to those three numbers, and let P be the determinate point within the triangle.

Then because three equal divisions are proposed, make AD equal to one third of AC, and draw DE parallel to AB. Likewise make BF equal to one third of BA, and parallel to BC draw FG, the point of intersection P of these parallels is the determinate point. To this point, therefore, from the three angles let CP, AP and BP be drawn; these will be the lines required. (*For demonstration see page* 167, *vol.* iii.)

REMARK. Had AB been bisected, a line from C to the point of bisection would have passed through P, and a line from A or B to the middle of the opposite side, by intersecting the line from C, would have determined P. It is therefore evident that PE is equal to PD, and each equal to $\frac{1}{3}$ of AB, that is, equal one third of 112·65 = 37·55. Consequently CD $= \frac{2}{3}(120·05) = 80·03$, and CE $= \frac{2}{3}(112·05) = 74·7$. Likewise the angle CDE ($= \angle$ CAB) = 57° .. 28′, and the angle CED ($= \angle$ CBA) = 64° .. 35′. Moreover EP = 37·55, EB = 37·35, and the angle BEP = the supplement of the

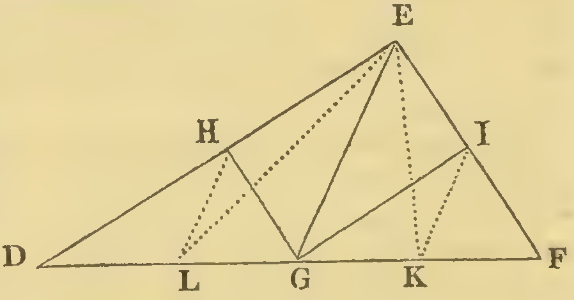
(Key to Vol. III. page 172.)

angle ABC $= 115° .. 25'$. Also AD $= 40{\cdot}015$ and the angle ADP$=$ $122° .. 32'$ the supplement of $57° .. 28'$ the angle BAC or EDC. Here, then, are three separate triangles, in each of which are given two sides and the included angle to find the third side. Hence CP $= 67{\cdot}693$, BP $= 63{\cdot}065$, and AP $= 68{\cdot}021$. *Which were to be determined.*



Ex. 7. $\sqrt{28^2+45^2} = \sqrt{2809} = 53 =$ the hypothenuse. Construct, therefore, a triangle of which the three sides are 28, 45, and 53, which let be DEF. Bisect DF in G, DE in H, and EF in I. Join GH, GE, GI, the lines GH, GE, and GI, are the lines required. For G is the middle of the hypothenuse, and the four triangles are evidently equal. [*Eucl.* i. 37.] It is likewise manifest that GI is parallel to DE, and that GH is parallel to EF. Consequently GH$=14$, and GI$=22\frac{1}{2}$. Lastly, GE $= \sqrt{\text{GH}^2+\text{GI}^2} = \sqrt{14^2+22{\cdot}5^2} = \sqrt{702{\cdot}25} = 26{\cdot}5$. Wherefore 14, 26·5, and 22·5 *are the three lines required.*

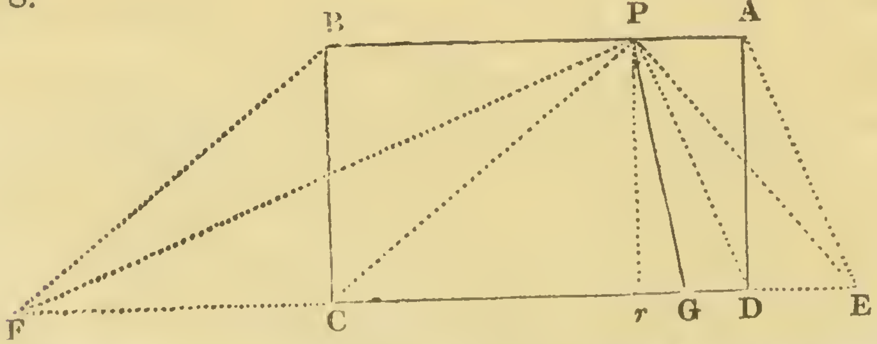
UNIVERSALLY.



Bisect DF in G, DG in L, and GF in K. Draw EG; and parallel to EG, through the points K and L, draw KI and LH. Join GI and GH, then DHG, HGE, EGI and IGF, are the four triangles required, and in all respects *as before.*

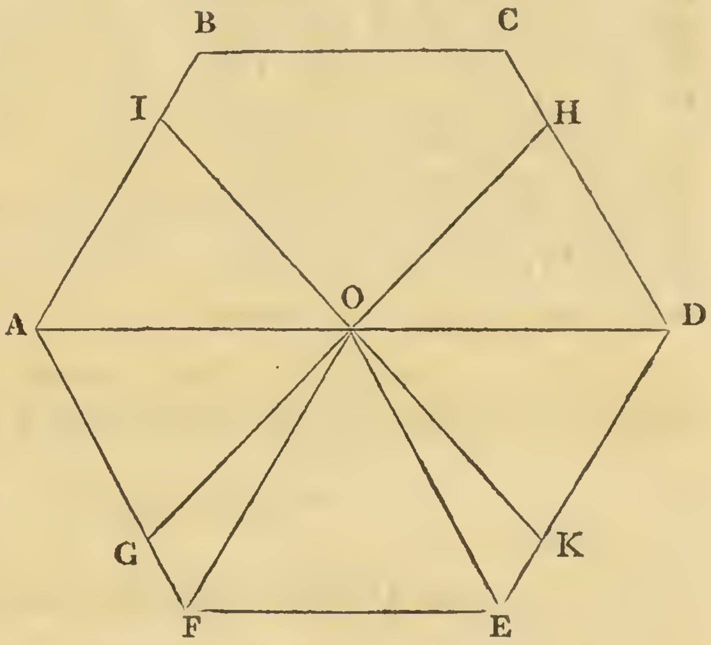
(Page 173.)

Ex. 8.



Construct the given rectangle, ABCD, and let P be the point in the longest side $\frac{4}{15}$ of AB from A. Join PC and PD, and through the point B draw BF parallel to PC, and through the point A draw AE parallel to PD: also produce CD both ways to E and F. Take EG equal to one fifth part of EF, and connect PG; GP is the line required. (*See Demonstration, page* 168, *vol.* iii.) Because EF is double of DC, $EG = 6$, and $GD = 2$. Through P, therefore, draw Pr parallel to AD or to BC, $rG = 2$. Consequently $PG = \sqrt{Pr^2 + rG^2} = \sqrt{9^2 + 2^2} = \sqrt{85} = 9{\cdot}2195445$. Q. E. I.

Ex. 9.



Let ABCDEF be the given hexagon, and O the center of the circumscribing circle. Draw any two lines as GH, IK, to represent the lines sought; also join OF, OE, OD, and OA. It is evident (since by the hypothesis OG and OK are equal) that, FG and EK must be equal.

If, therefore, a be substituted for $AF = FE = ED = 12$ by the question, x for $FG = EK$, and am for the area of the equilateral triangle FOE, the area of the triangle GOF will be truly expressed by xm, and the area of the triangle AOG by $(a - x)\,m$. [*Eucl.* B. vi. *Prop.* 1.]

But the two triangles GOF, KOE are in all respects equal, and the triangles AOG, HOD, DOK, are likewise equal and similar; consequently the whole area GOKEF $= am + 2xm$, and the whole area KOHD $= 2(a - x)m = 2am - 2xm$. Now by the question these two whole areas are equal; wherefore $am + 2xm = 2am - 2xm$, that is, $a + 2x = 2a - 2x$. Hence $x = \frac{1}{4}a = 3$, and the angle GOF $= 13° \,..\, 53' \,..\, 52''{\cdot}8$.

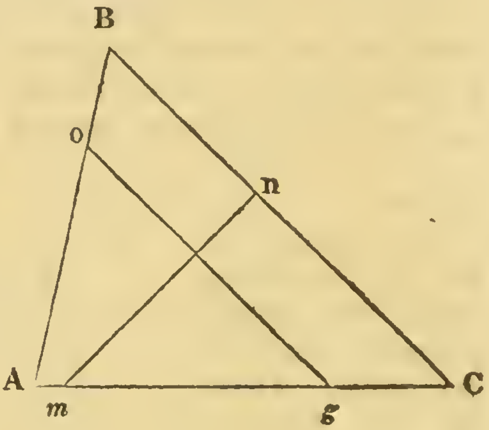
Also by analogy,

As Sin. 13° .. 53′ .. 52″·8 (∠GOF) Log. ar. co. −10·6194376
: Sin. 60° (∠OFG) - - - - - - - - - - - - Log. 9·9375306
: : 3, (side FG) - - - - - - - - - - - - - - Log. 0·4771213

: 10·81656 (*side* OG) - - - - - - - - - - - *Log.* 1·0340895

(Key to Vol. III. page 173.)

But $2 \times 10{\cdot}81656 = 21{\cdot}63312 =$ GH $=$ IK, which being the *only equal* lines that can possibly divide the hexagon into four equal parts, *are necessarily the shortest possible.* Q. E. I.



Ex. 10. Let ABC be the proposed triangle, AB$=5$, AC$=6$, and BC$=7$.

Draw *mn* at right angles to BC, and dividing the triangle ABC into two equal parts, by Problem I. Case 3, page 161, vol. iii.; and draw *o g* parallel to BC, dividing the triangle ABC into two equal parts, by Prob. I. as above, Case 2. So shall *m n*, *o g* be the two lines required.

The demonstration is obvious.

*** *This example admits of three answers, since it is immaterial whether* mn *be drawn at right angles to* BC, *to* AB, *or to* AC.

SOLUTION OF EQUATIONS.

(Page 186)

Ex. 4. Because the 1st term of an equation is the unknown quantity raised to the power denoted by the number of roots, the 1*st term* of the equation to be produced must be x^2.

And because the 2nd term will contain the unknown quantity raised to a power less than the former by *unity*, with a co-efficient equal to the sum of the roots taken with contrary signs, it follows that $(-\frac{3}{4}-\frac{4}{5})x = -\frac{31}{20}x$ is the *second term* of the equation sought. Also, because the unknown quantity raised to a power less by 2 than that of the 1st term, with a co-efficient equal to the sum of all the products that can arise by multiplying all the roots of the equation two and two, will constitute the third term of the equation, $(\frac{3}{4}\times\frac{4}{5})=\frac{12}{20}=\frac{3}{5}$ must be the *third and last term* of the equation required.

Therefore $x^2 - \frac{31}{20}x + \frac{3}{5} = 0$. Ans.

(Key to Vol. III. page 187.)

Ex. 5. It is evident by the last example that the leading term will be x^3.

Now the sum of 2, 5, and -3 taken with contrary signs is -4; consequently the *2nd term* must be $-4x^2$.

Likewise the sum of (2×5), (2×-3), and (5×-3) is $10-6-15=-11$; therefore $-11x$ is the *3rd term* of the equation.

And since the 4th term will consist of the unknown quantity raised to a power less by 3 than that of the 1st term, with a co-efficient equal to the sum of all the possible trinary products of the roots taken with contrary signs, it follows that $(-2\times-5\times+3)x^0=30$ is the *4th and last term* of the equation proposed.

Hence, when connected, $x^3-4x^2-11x+30=0$. Ans.

Ex. 6. It is manifest by examples 4th and 5th, [*and Theor.* i. *page* 176] that the powers of the unknown quantity in the several terms will be $x^4 \ldots . x^3 \ldots . x^2 \ldots . x^1 \ldots . x^0$, and that the sign and co-efficient of the leading term will be $+1$.

Now for the co-efficient of the second term *(equal to the sum of all the roots taken with contrary signs)* it will be, $-1-4+5-6=-6$ the co-efficient of the *2nd term.*

And for the co-efficient of the 3d term, *(equal to the sum of all the binary products of the roots,)* it will be
$(1\times4)+(1\times-5)+(1\times6)+(4\times-5)+(4\times6)+(-5\times6)=4-5+6-20+24-30=-21$ the co-efficient of the *3d term.*

Next for the co-efficient of the 4th term *(equal to the sum of all the trinary products of the roots taken with contrary signs)* it will be $(-1\times-4\times+5)+(-1\times-4\times-6)+(-1\times+5\times-6)+(-4\times+5\times-6)=20-24+30+120=+146$ the co-efficient of the *4th term.*

Lastly, since the co-efficient of the 5th term will be the continued product of all the roots taken with contrary signs, it will evidently be, $-1\times-4\times+5\times-6=-120$ the co-efficient of *that term.*

Hence by uniting the co-efficients with the powers of the unknown quantity, and connecting the terms, there arises $x^4-6x^3-21x^2+146x-120=0$ *the equation required.*

Ex. 7. Here $p=347$, $q=22110$ and the equation agrees with the 1*st* FORM.

Now taking the tang. $A=\frac{2}{p}\sqrt{q}$, the two roots of the proposed equation are $x=+$ tang. $\frac{1}{2}A\sqrt{q}$, and $x=-$ cot. $\frac{1}{2}A\sqrt{q}$.

(Key to Vol. III. page 187.)

In Logarithms.

$\frac{1}{2}$ log. of 22110 - - - - - - is 2·1722943=log. $\sqrt{q}$.
log. of 2 - - - - - - is 0·3010300
Arith. com. log. 347 - - - - - - is $-$3·4596705

The sum of these 3 logarithms is $-$1·9329948=log. tang. 40° .. 35′ .. 50″·94 to radius 1.

Hence

$\frac{1}{2}$A=20° .. 17′ .. 55″·47, of which the log. tang. is $-$1·5680682
Also log. $\sqrt{q}$ *(as already found)* - - - - is 2·1722943

The log. sum - - - is 1·7403625 the logarithm of 55 one of the values of x as a root of the equation.

Again,

Log. cot. 20° .. 17′ .. 55″·47 is 0·4319318
Log. $\sqrt{q}$ - - - - - - - - - - is 2·1722943

The log. sum - - - is 2·6042261 the log. of 402, therefore $-$402 is the value of x in the *other* root of the given equation. Consequently 55 or $-$402. Ans.

Ex. 8. Here $p=\frac{55}{12}$, $q=\frac{325}{6}$, and the equation agrees with the 2*nd* FORM. Wherefore

If tang. A $=\frac{2}{p}\sqrt{q}$, the two values of x in the roots of the given equation, are $-$tang. $\frac{1}{2}$A $\sqrt{q}$, and $+$ cot. $\frac{1}{2}$A $\sqrt{q}$.

In Logarithms.

Log. of 325 - - - is 2·5118834
Log. of 6 - - - is 0·7781513

$\frac{1}{2}$ *the log. diff.* - - is 0·8668660 the log. $\sqrt{q}$.
Log. of 24 - - - is 1·3802112
Ar. co. log. of 55 is $-$2·2596373

The sum of the 3 last logs. - - - - - is } 0·5067145 *the log. tang.* 72° .. 42′ .. 17″=A.

Or $\frac{1}{2}$A=36° .. 21′ .. 8″$\frac{1}{2}$ of which the log. tang. is $-$1·8668660
But the log. $\sqrt{q}$ *(already found)* - - - - is ·8668660

The sum of these two logarithms is 0·7337320 the logarithm of $5·41666\dot{6}=\frac{65}{12}$

Therefore $-\frac{65}{12}=x$ in *one* of the roots of the equation.

Again,

Log. cot. 36° .. 21′ .. 8″$\frac{1}{2}$, ($\frac{1}{2}$A) - is 0·1331340
Also log. $\sqrt{q}$ *(as above)* - - - is 0·8668660

The log. sum - - - - - - - - - is 1·0000000 = log. of 10.

Consequently 10, and $-\frac{65}{12}$ are the *two* values of x in the proposed equation. Q. E. I.

Ex. 9. In this example $p=\frac{264}{25}$, $q=\frac{695}{25}$, and the equation is of the 4*th* FORM. Hence,

If sin. A $=\frac{2}{p}\sqrt{q}=\frac{50}{264}\sqrt{q}$, x in *one* root is equal to tang. $\frac{1}{2}$A $\sqrt{q}$, and in the *other* root equal to the cot. $\frac{1}{2}$A $\sqrt{q}$.

In Logarithms.

Log. of 695 - - is 2·8419848
Log. of 25 - - is 1·3979400

$\frac{1}{2}$ *the log. diff.* - is 0·7220224 = log. $\sqrt{q}$.
Log. of 50 - - - is 1·6989700
Ar. co. log. 264 is −3·5783961

Sum of the 3 last logs. is −1·9993885 = log. sin. 86° .. 57′ .. 36″ = A.

Wherefore

$\frac{1}{2}$A = 43° .. 28′ .. 48″, of which the log. tang. is −1·9769470
But log. $\sqrt{q}$ *(already found)* - - - - - - is 0·7220224

The log. sum is 0·6989694 = the log. of $5=x$ in *one* of the roots of the equation.

Again,

Log. cot. of 43° .. 28′ .. 48″ is 0·0230530
And log. $\sqrt{q}$ *(as above)* 0·7220224

The log. sum is 0·7450754 the log. of 5·56 $=\frac{139}{25}$. Therefore 5 and 5·56 are the *two* values of x in the given equation. Q. E. I.

Ex. 10. The proposed equation is of the 4*th* FORM, wherein $p=$ 24113, and $q=$481860.

Taking, therefore, sin. A $=\frac{2}{p}\sqrt{q}$ the two values of x are the tangent and cotangent of A $\sqrt{q}$.

(Key to Vol. III. page 187.)

In Logarithms.

$$\begin{array}{rl} \tfrac{1}{2}\,\text{log. } 481860 = & 2{\cdot}8414604 = \text{log. } \sqrt{q} \\ \text{Log. of } 2 = & 2{\cdot}3010300 \\ \text{Ar. co. log. } 24113 = & -5{\cdot}6177488 \\ \hline \end{array}$$

The log. sum $= -2{\cdot}7602392$ the log. sin. 3°..18′..2″..24‴ $=\mathrm{A}$.

$$\left.\begin{array}{r}\text{Consequently } \tfrac{1}{2}\,\mathrm{A} = 1^\circ..39'..1''..12''' \text{ of} \\ \text{which the log. tang. is - - - - -}\end{array}\right\} -2{\cdot}4595690$$

The log. $\sqrt{q}$ *(already found)* is $2{\cdot}8414604$

The log. sum is $1{\cdot}3010294$ the logarithm of 20.

Again,

Log. cot. of 1° .. 39′ .. 1″ .. 12‴ is $+1{\cdot}5404310$

And log. $\sqrt{q}$ *(as above)* is $2{\cdot}8414604$

The log. sum is $4{\cdot}3818914$ the log. of 24093. Wherefore 20 and 24093 are the *two values required.*

Ex. 11. Here $p=3$, $q=1$, and the equation agrees with the *2nd* Form for Cubic equations; consequently,

If $\sin.\ \mathrm{B} = \dfrac{p}{3q}\, 2\sqrt{\tfrac{1}{3}p}$, $\text{tang. } \mathrm{A} = \sqrt[3]{\ }\text{tan. } \tfrac{1}{2}\mathrm{B}$.

Then $x = \mp \text{co. sec. } 2\mathrm{A} \times 2\sqrt{\tfrac{1}{3}p}$.

But $\dfrac{\frac{1}{3}p}{q} \times 2\sqrt{\tfrac{1}{3}p} = \dfrac{1}{1} \times 2\sqrt{1} = 2 = \sin.\ \mathrm{B}$ and evidently greater than *Unity:* wherefore taking co. sec. of $3\mathrm{A} = \dfrac{\frac{1}{3}p}{q} \times 2\sqrt{\tfrac{1}{3}p}$ $= 2 = \text{co. sec. } 30^\circ$, it will be, $\mathrm{A} = 10^\circ$. And

$$x = \pm \sin.\ \mathrm{A} \times 2\sqrt{\tfrac{1}{3}p} = \pm \sin.\ 10^\circ \times 2.$$

$$x = \pm \sin.\,(60^\circ - \mathrm{A})\, 2\sqrt{\tfrac{1}{3}p} = \pm \sin.\,(60^\circ \mp 10^\circ) \times 2 = \pm 2 \times \begin{Bmatrix} 50^\circ \\ 70^\circ \end{Bmatrix}$$

And

$$x = \pm \sin.\,(60^\circ + \mathrm{A})\, 2\sqrt{\tfrac{1}{3}p} = \pm \sin.\,(60^\circ \pm 10^\circ) \times 2 = \pm 2 \times \begin{Bmatrix} 70^\circ \\ 50^\circ \end{Bmatrix}$$

Again, [*Theor.* 2. *page* 178. *vol.* iii.] it is evident by the *variation* and *permanency* of the signs of the proposed equation that one of the roots, *only*, is *positive* and two *negative*, and that the double sine of 70° is the positive root. Consequently

(Key to Vol. III. page 187.)

$$x=\left\{\begin{array}{lcl} +1\cdot8793852 & = & \sin.\,70^\circ \\ -1\cdot5320888 & = & -\sin.\,50^\circ \\ -\ \ \cdot3472964 & = & -\sin.\,10^\circ \end{array}\right\}\text{to Radius 2.}$$

Ex. 12. Here $p=1$, and $q=6$; wherefore $\frac{\frac{1}{3}p}{q}=\frac{1}{18}$, and $2\sqrt{\frac{1}{3}p}=\sqrt{\frac{12}{9}}=\frac{2}{3}\sqrt{3}$. Hence $\frac{\frac{1}{3}p}{q}\times 2\sqrt{\frac{1}{3}p}=\frac{1}{27}\sqrt{3}$.

In Logarithms.

½ log. of 3 = 0·2385606
log. of 1 = 0·0
Ar. co. log. of 27 = −2·5686362

The log. sum = −2·8071968 the log. sin. $\left\{\begin{array}{r} 3^\circ..40'..41'' \\ 176^\circ..19'..19'' \end{array}\right\}=$ B

Or, $\frac{1}{2}$B $=\left\{\begin{array}{l} 1^\circ..50'..20''\tfrac{1}{2} \\ 88^\circ..\ 9'..39''\tfrac{1}{2} \end{array}\right\}$

But log. tang. A = log. $\sqrt[3]{\ }$ tang. 88°..9′..39″½ = 0·4977964 the log. tang. of 72°..22′..4″, which doubled = 2A = 144°..44′..8″.

And co. sec. 144°..44′..8″ = co. sec. 35°..15′..52″ = sec. 54°..44′..8″.

Hence (co. sec. 2A × $2\sqrt{\frac{1}{3}p}=$) $x=$

$$\frac{\ }{3}\sqrt{3}\times\left\{\begin{array}{lrrr} \text{co. sec.} & 144^\circ & ..\ 44' & ..\ 8'' \\ \text{co. sec.} & 35^\circ & ..\ 15' & ..\ 52'' \\ \text{sec.} & 54^\circ & ..\ 44' & ..\ 8'' \end{array}\right\}\text{Ans.*}$$

* *In this Example the Answer given in the* COURSE *differs* 12 SECONDS *from the Answer in the Key.*

Ex. 13. Given $25x^3+75x-46=0$: that is, divided by 25,

$$x^3+3x-\frac{46}{25}=0.$$

Here $p=3$, $q=\frac{46}{25}$, and the equation is of the 1*st* CUBIC FORM. Wherefore

$$\text{Tang. B.}=\frac{\frac{1}{3}p}{q}\times 2\sqrt{\tfrac{1}{3}p}=\frac{25}{46}\times 2=\frac{50}{46}.$$

In Logarithms.

Log. of 50 is 1·6989700
Ar. co. log. 46 is −2·3372422

The log. sum is 0·0362122 = log. tan. B.

$$\text{That is, } \mathrm{B} = \left\{ \begin{matrix} 47^\circ..23'..\;9''\tfrac{2}{5},\text{ or} \\ 132^\circ..36'..50''\tfrac{3}{5} \end{matrix} \right\}$$

Taking the latter value,

$\frac{1}{2}\mathrm{B} = 66^\circ..18'..25''\cdot3$ of which the log. tang. is 0·3576517

But log. tang. $\mathrm{A} = \log. \sqrt[3]{}$ tang. $\frac{1}{2}\mathrm{B} = 0\cdot1192172$ the log. tang. of $52^\circ..46'..1''\frac{3}{4}$ *nearly*. Consequently

$2\mathrm{A} = 105^\circ..32'..3''\frac{1}{2}$ *nearly*, or (the supplement of that arc $=$) $74^\circ..27'..56''\frac{1}{2}$.

$$\text{Hence } x\left(=\text{cot. } 2\mathrm{A} \times 2\sqrt{\tfrac{1}{3}p}\right) = 2\text{ cot.} \left\{ \begin{matrix} 74^\circ..27'..56''\tfrac{1}{2} \\ 105^\circ..32'..\;3''\tfrac{1}{2} \end{matrix} \right\} \text{ Ans.*}$$

* Differing 8·5 seconds from the Answer given with the Question.

Ex. 14. Because $x^4 - 8x^3 - 12x^2 + 84x - 63 = 0$, it follows that $x^4 - 8x^3 = 12x^2 - 84x + 63$. Hence, by adding $16x^2$ to both sides of the equation,

$x^4 - 8x^3 + 16x^2 = 28x^2 - 84x + 63 = 7(4x^2 - 12x + 9)$, Or $(x^2 - 4x)^2 = 7(2x - 3)^2$; that is, $x^2 - 4x = \pm\sqrt{7} \times (2x - 3)$.

Consequently $x^2 - (4 \pm 2\sqrt{7})x = \mp 3\sqrt{7}$ a quadratic equation; which solved gives $x = 2 + \sqrt{7} \pm \sqrt{11 + \sqrt{7}}$. Ans.

Ex. 15. If $548x^2$ be added to both sides of the given equation, it becomes,

$$x^4 + 36x^3 + 148x^2 - 3168x + 7744 = 548x^2.$$

And extracting the square root on both sides,

$x^2 + 18x - 88 = \pm x\sqrt{548}$. That is,

$x^2 + (18 \mp \sqrt{548})x = 88$; and solving this quadratic equation, $x = 12\cdot46766$ *very nearly*. Ans.*

* *We cannot help remarking the great disparity between our Answer and that of Dr. Hutton, in the present Example.*

Ex. 16. Here, as in Ex. 15, the co-efficient of the 4th term divided by that of the second gives the square root of the last term for the quotient; therefore, adding $256x^2$ to both sides of the biquadratic, there arises

$$x^4 + 24x^3 + 142x^2 - 24x + 1 = 256x^2.$$

And by extracting the square root on both sides,

$$x^2 + 12x - 1 = \pm 16x. \quad \text{Hence}$$

$$x^2 + \left\{ \begin{matrix} -4 \\ +28 \end{matrix} \right\} x = 1.$$ Wherefore

$$x = \left\{ \begin{matrix} 2 \pm \sqrt{5} \\ -14 \pm \sqrt{197} \end{matrix} \right\}$$ Ans.

Ex. 17. Adapting the given equation to Euler's Rule, it will be $x^4 - 0x^2 - 12x - 5 = 0$. Hence the roots are $1 \pm \sqrt{2}$ and $-1 \pm 2\sqrt{-1}$.

Ex. 18. Adding x^2 to both sides of the given equation, there arises $x^4 - 12x^3 + 48x^2 - 72x + 36 = x^2$, two perfect squares. And, by extraction of the square root, $x^2 - 6x + 6 = x$, or $x^2 - 7x = -6$.

But the two roots of this quadratic *(and consequently two roots of the biquadratic)* are $3{\cdot}5 \pm 2{\cdot}5$, that is, 6 and 1.

Now the other two roots of the proposed equation admit of being found in a variety of ways; for it is evident, by the alternation of the signs, that they are positive, and by the co-efficient (-12) of the second term, that their sum is 6.

Let them be found, however, in the following manner:

Divide the given equation by $x - 6$, and it is depressed to the cubic equation $x^3 - 6x^2 + 11x - 6 = 0$.

In like manner divide the given equation by $x - 1$, and it is depressed to the cubic equation $x^3 - 11x^2 + 36x - 36 = 0$.

Putting these two cubic equations equal, and reducing, the result is $x^2 - 5x = -6$.

Now the two roots of this quadratic, *(and consequently the other two roots of the biquadratic)*, are $2{\cdot}5 \pm {\cdot}5$, that is, 3 and 2.

By which it is plain, that the four roots required are 1, 2, 3, and 6. Q. E. I.

Ex. 19. If the given equation be divided $x + a$, it will be depressed to $x^4 - 6ax^3 - 74a^2x^2 + 6a^3x + a^4 = 0$, a biquadratic equation. Whence it follows that $-a$ is one of the roots of the sursolid equation.

Let $81a^2$ be now added to both sides of the new equation, and there will arise $x^4 - 6ax^3 + 7a^2x^2 + 6a^3x + a^4 = 81a^2$, two perfect squares; and by extraction of the square root, $x^2 - 3ax - a^2 = 9a$, or $x^2 - 3ax = a^2 + 9a$.

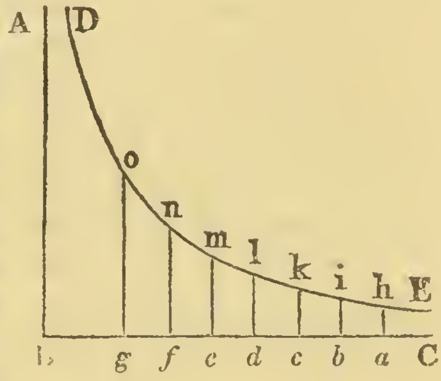
But the two roots of this quadratic *(and consequently two roots of the biquadratic,* and also of the proposed equation,) are $\frac{3}{2}a + \sqrt{\frac{13}{4}a^2 + 9a}$, and $\frac{3}{2}a - \sqrt{\frac{13}{4}a^2 + 9a}$, by which the remaining two roots are known. Or the biquadratic may be transformed into an equation that shall want the 2nd term, (and to which Euler's rule will be applicable), by assuming

(Key to Vol. III. page 187.)

$m = -\frac{3}{2}a$, and substituting $z - m$ (that is, $z + \frac{3}{2}a$) for x. Or, by the rule for the solution of biquadratic equations, page 183, vol. iii. of the Course, the biquadratic above found being of the form $x^4 + 2px^3 = qx^2 + rx + s$, where $p = -3a$, $q = 74a^2$, $r = -6a^3$, and $s = -a^4$, the four roots are obtained from two quadratic equations. These roots, by whatever method determined, are surd quantities, more or less complicated, and in the simplest form are $-3a + a\sqrt{10}$, $-3a - a\sqrt{10}$, $6a + a\sqrt{37}$, and $6a - a\sqrt{37}$.

EQUATIONS TO CURVES.

(Page 205.)



Ex. 2. If in the right line BC there be taken a point a, and if the points b, c, d, e, f, g, be assumed such that ab, ac, ad, ae, af, and ag, be in arithmetical progression, then if ah, bi, ck, dl, &c. be in geometrical progression, and perpendicular ordinates at those points, the curve D $o\,n\,m\,l\,k\,i\,h$ E passing through the extremities of the ordinates, is the logarithmic curve, BC an asymptote to the curve, and AB (being parallel to, and greater by the least assignable magnitude than, the greatest ordinate,) is the other asymptote.* Because the abscissæ de, df, and dg, are in arithmetical progression, and their ordinates em, fn, and go, in geometrical progression, it follows that, when $dl = 1$, and $em = \alpha$, then $fn = \alpha^2$, $go = \alpha^3$, &c.

Wherefore, if x represent any part whatever of the arithmetical series ascending, measured from d, the corresponding ordinate is α^x; that is, $y = \alpha^x$, *α being the number of which the logarithm is* 1 *in the system of logarithms represented by the curve.*

---

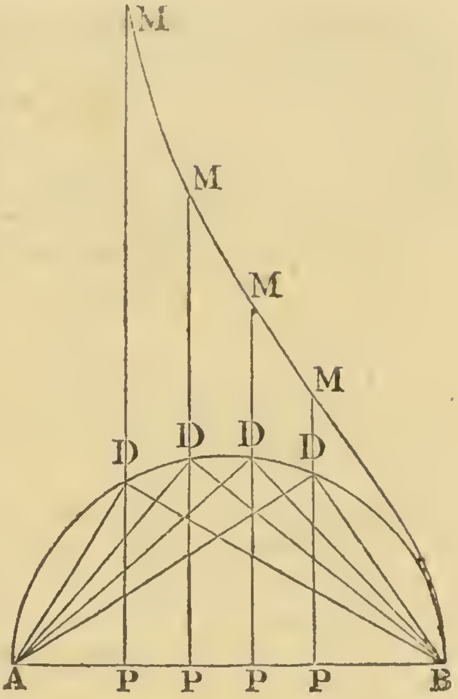
* For the square of any term of a geometrical series is equal to the rectangle of any two terms equidistant from it; consequently the square of dl is equal to $og \times ah$; but og increases as ah decreases; yet, it is plain, ah can never be equal to 0, nor og infinite. Therefore BC continually approaches the curve, and when og is inconceivably near infinite by ah being inconceivably near nothing, the distance of og from AB is less than the least assignable magnitude, and AB is an asymptote to the curve.

(Key to Vol. III. page 205.)

Ex. 3. Let the curve be MB, and the generating semicircle ADB; also let AP be any absciss whatever, end PM the corresponding ordinate; then by the properties of the curve, As AP : PD : : AB : PM.

But the angle ADB is a right angle, [*Eucl.* iii. 31.] and it is As AP : PD : : PD : PB. [*Eucl.* vi. 8.]

If, therefore, d be put for AB, the value of PD is $\sqrt{dx-x^2}$, and As $x : \sqrt{dx-x^2} :: d : y$. That is $xy = d\sqrt{dx-x^2}$, and $y = \dfrac{d\sqrt{dx-x^2}}{x} = d\sqrt{\dfrac{dx-x^2}{x^2}} = d\sqrt{\dfrac{d-x}{x}}$. Q. E. I.



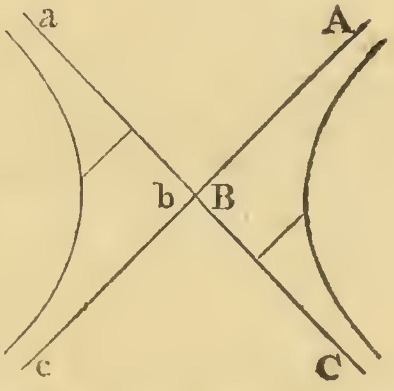
LOCI OF EQUATIONS.

(Page 208.)

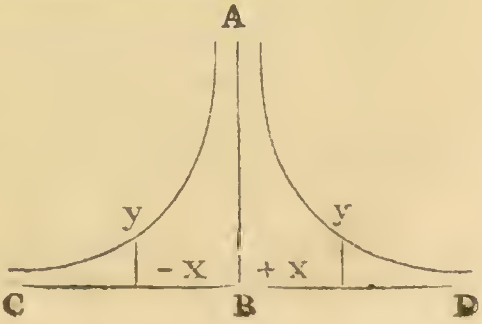
Ex. 4. First let n be some odd number, and x positive, so shall $y = \dfrac{a^{n+1}}{x^n}$.

Next let n be some odd number and x negative, so shall $y = -\dfrac{a^{n+1}}{x^n}$.

By which it is evident that the *loci* of the curve will be in the vertical angles ABC, *abc*, which situation corresponds to the common hyperbola, AC, and *ac* being asymptotes to the curve.



Again let n be any even number whatever; it is plain, since x^n must be positive, that y must also be positive; and in this case the curve will fall in the adjacent angles ABC and ABD, having AB, BC, and AB, BD for asymptotes.



(Key to Vol. III. page 208.)

Whereby the principal properties of the curve are apparent. Q. E. D.

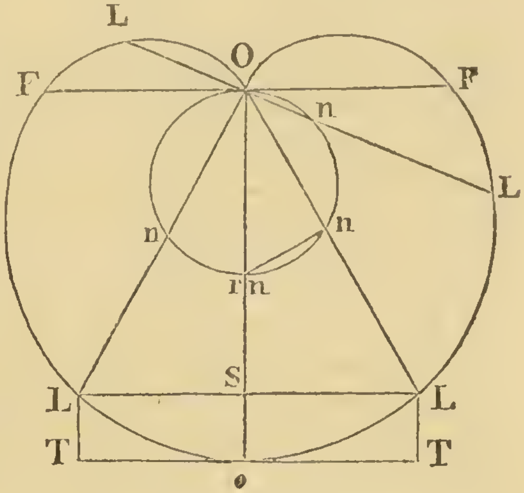
Ex. 5. Here $y=\frac{bc+bx}{a+c+x}$, by which it appears that the ordinate can meet the curve in *one point only*, there being a single corresponding value of y for each value of x. Now when $x=0$, $y=\frac{bc}{a+c}$, consequently the curve does not pass through the point from which the values of x are measured.

Again it is manifest, that the quantity $a+c+x$ is greater than the quantity $c+x$, therefore, although y must increase when x increases, yet y can never be equal to b, except when x becomes infinite; in which case since a and c *(finite quantities,)* bear no comparison with x *(an infinite quantity,)* y may be truly expressed by $b\times\frac{x}{x}$, that is, by b. The asymptote, therefore, will touch the curve at an infinite distance. But if x be supposed negative, then $y=b\times\frac{c-x}{a+c-x}$ and when $x=c$, $y=b\times\frac{c-c}{a+c-c}=\frac{0}{a}=0$. Wherefore B is a point in the curve as often as AB $=c$.

On the other hand, if c be less than x, it is plain that $c-x$ is negative, for which reason the ordinate to the curve will be negative until x become equal to $a+c$; at which situation $y=b\times\frac{-a}{o}$, that is *infinity;* and, therefore, x infinite. Consequently if any point M be assumed in the straight line on which x is measured, so that $a+c=$ AM, and if through M there be drawn a right line parallel to an ordinate, that right line shall be an asymptote to the curve.

Lastly, if a and c together be less than x, (x being still negative), then $c-x$ and $a+c-x$ are both negative; but y positive, because b multiplying a negative quantity divided by a negative quantity is positive: and since, in this case, $x-c$ must be always greater than $x-a-c$, it follows that y will always exceed b. Hence the several *loci* of the curve are confined to the angular space vertical to that in which they fall in the first supposition. The ratio, however, of $x-c$ to $x-a-c$, approaches more and more to the ratio of

1 to 1 as x increases, and is the ratio of 1 to 1 when x is infinite. This signifies that the curve continually approaches to its asymptote in this angular space likewise, and must therefore be the common hyperbola. To prove which, let ab be added to both sides of the given equation, then $y(a+c+x)+ab=b(a+c+x)$, and $ab=(b-y)\times(a+c+x)$. Whereby the truth of the inference is manifest. Q. E. D.



Ex. 6. If O*o* be assumed equal to a the line of the abscisscæ, and if SL be an ordinate at right angles to this line, it is manifest, by the equation of the curve, that $x=a$, when $y=0$; therefore, the curve must pass through the point *o*, and have T*o*T a line parallel to SL, for a tangent at that point.

It likewise appears by the equation of the curve, that when x vanishes, $y=\frac{1}{2}a$; wherefore, if OF be drawn from the extremity O of the line O*o*, equal to $\frac{1}{2}a$, and parallel to TT, F will be a point of the curve on either side of the line of the abcisscæ.

Finding in like manner other values of y, the several points L, L, L, L, are determined to be loci of the equation, such that, if a circle be described on the half of a, nL is always equal to the radius of that circle. For, putting OL $=z=\sqrt{x^2+y^2}$, and the angle *o*OL $=v$, it will be, As z : rad. : : x : cos. v, that is, taking *Unity* for radius, $x=\cos. v$. Consequently $a^2z^2=4(z^2-\frac{1}{2}az\times\cos. v)^2$; or by extracting the square root on both sides, dividing by z, and transposing, $z=\frac{1}{2}(a+a.\cos. v.)$

Now because O$r=\frac{1}{2}a$, if rn be drawn at right angles to OL, the proportion is,

As Or $(=\frac{1}{2}a)$: rad. (1) : : On : cos. v, that is, O$n=\frac{1}{2}a.\cos. v$.

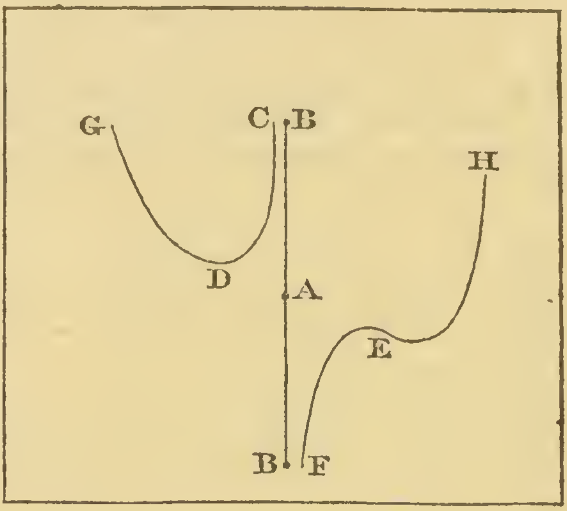
But $z=\frac{1}{2}(a+a.\cos. v)=\frac{1}{2}a+On=OL=$(*because* O$r=\frac{1}{2}a$,) O$r+OL-n$L.

Hence Or $(=r$O$)=n$L; and nL is every where equal to Or; FF, or LL, is equal to 2Or; OL$=$O$r\pm$On; and n always bisects LL.

⁂ *Many other properties of the Cardioide may be seen in the Philos. Trans. for* 1741.

(Key to Vol. III. page 208.)

Ex. 7. Here $y = \frac{ax^3 + bx^2 + cx + d}{x}$, which is infinite when $x=0$. And when x is infinite y is also infinite, because equal to infinity *plus* a finite quantity *plus* nothing. Again, when $x=$*unity* the value of y is $a+b+c+d$; or if x be any arbitrary quantity, and either one or more of the co-efficients $=0$, then the corresponding term or terms of the equation must vanish or become wanting. By which it appears, that the figure has four infinite branches or legs; and that since in two of them the tangent to the curve coincides with the asymptote when the point of contact is at an infinite distance, but in the other two recedes *in infinitum* and is no where found, necessarily the former are diverging hyperbolic legs in contrary directions, as CD and EF about the asymptote AB, and the latter converging parabolic legs as GD and HE meeting the hyperbolic branches in D and E. It is likewise evident that the simplest equation to the curve is when the rectangle under the absciss and ordinate is equal to x^3+a^3, for the loci of the curve are in this case easily determinable by a common parabola whose absciss is ax^2+bx+c, and an hyperbola having its absciss $= d \div x$, for y will be at all times equal to the *sum* or *difference* of the correspondent ordinates of those two conic sections.



Ex. 8. First for the Cissoid. Because the equation of the curve is $x^3 = y^2 \times (d-x)$, and because it is plain that $d-x$ decreases as x increases, it follows that y, and consequently y^2, must continually increase; otherwise $(d-x)y^2$ could not be equal to x^3.

Again, it is manifest that x can never equal d, for in that event x^3 would equal 0, which is absurd; wherefore y or PM may approach within less than the least assignable magnitude of BX, but can never coincide with it. *That is,* BX *is an asymptote*[*] *to the curve* AMQ.

Next for the Witch. It is evident by the equation of the curve that the value of $d-x$, and consequently that of

---

[*] The curve AMQ bisects the curve ATB, and the cissoidal space under the diameter AB, the asymptote BX and the curve AMQ infinitely produced, is six times the area of the generating semicircle.

(Key to Vol. III. page 208.)

$d\sqrt{\frac{d-x}{x}}$, must decrease by the increase of x. Now when $x=d$, y vanishes; but when $x=0$, y is infinite, since $d\sqrt{\frac{d-0}{0}}$ is infinite. If, therefore, a tangent be drawn to the generating semicircle at the point A, *that tangent will be an asymptote to the curve* MB, *inasmuch as* PM *may approach within less than the least assignable magnitude of the tangent, but can never coincide with it.*

FLUXIONS AND FLUENTS.

(Page 229.)

Ex. 5. Ans. $-\frac{3}{5}x^2\times(a^2-x^2)^{\frac{3}{2}}-\frac{4}{15}a^2$.

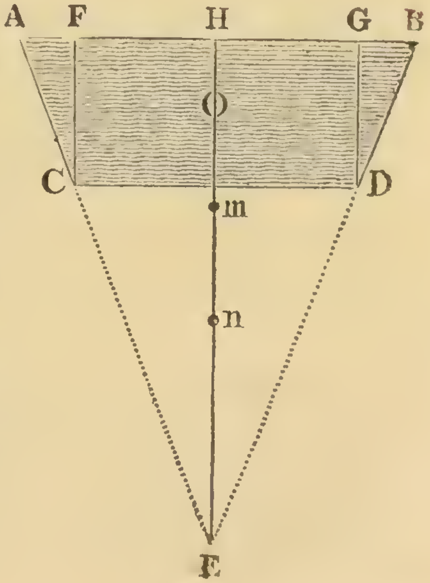
Ex. 6. Ans. $-2x^4\times(a^2-x^2)^{-\frac{1}{2}}-\frac{8a^2x^2+16a^4}{3}$.

Ex. 7. Ans. $\frac{\overline{a-x^n}|^{\frac{3}{2}}}{-\frac{7}{2}anx^{\frac{7}{2}n}}\times\overline{1+\frac{\frac{4}{5}ax^n+\frac{8}{15}x^{2n}}{a^2}}=$

$$-\frac{\overline{a+x^n}|^{\frac{3}{2}}\times\overline{30a^2+24ax^n+16x^{2n}}}{105na^3x^{\frac{7}{2}n}}.$$

LATERAL PRESSURE OF FLUIDS.

(Page 262.)



Ex. 3. Let ACDB be a perpendicular section of the canal, and let the sides AC and BD of the trapezium AD, meet, when produced, in E; also let EH be perpendicular to AB, and parallel to EH draw CF and DG. It will be, As AF : FC : : AH : HE ∵ 3 ft. : 8 ft. : : 10 ft. : $26\frac{2}{3}$ ft. = EH.

Now if o be assumed as the center of gravity of the trapezium, m as the center of gravity of the triangle ABE, and n that of the triangle CDE, then Hm [*Statics, art.* 229. *vol.* ii.] = $8\frac{8}{9}$ ft. and En = $12\frac{4}{9}$ ft. But, As trapez. AD : ΔCDE : : nm : mo ∵ $(26\frac{2}{3})^2-(18\frac{2}{3})^2 : (18\frac{2}{3})^2$: : $5\frac{1}{3}$ ft. : $5\frac{1}{8}$ ft. = mo.

(Key to Vol. III. page 262.)

Hence the depth of o below the surface of the water is $3\frac{55}{72}$ ft. which drawn into 136, the area of the dam-gate, the result is 511·9 *very nearly*.

But 511·9 cubic feet of water weigh 511900 oz. avoirdupois = the pressure = $31993\frac{3}{4}$ lb. = 14·28292 tons. Ans.

Otherwise and more elegantly by Mr. DOWLING'S Rule, as practised at the Academic Institution, Highgate.

To $\frac{1}{3}$ *the difference of the breadth at the surface and bottom in feet add the less breadth, and multiply the sum by half the square of the depth for the number of Cubic feet of water whereof the absolute weight shall be equal to the pressure required.*

$\frac{1}{3}$ the diff. of the breadths is 2 feet.
Add the less breadth - - - - 14 feet.

Sum 16 feet.
Multiply by $\frac{1}{2}$ the square of the depth = 32 square feet.

PRODUCT 512 cubic feet.

But the absolute weight of 512 cubic feet of water is 14·28571 &c. tons *the true pressure required.*

FINIS.

J. M'Creery, Printer,
Black-Horse-Court, London.

Prospectus

OF THE

ADVANTAGES ENJOYED BY THE PUPILS

AT THE

ACADEMIC INSTITUTION,

MANSION-HOUSE, HIGHGATE,

CONDUCTED BY

DANIEL DOWLING,

PROFESSOR OF THE MATHEMATICS; LECTURER IN NATURAL
PHILOSOPHY AND ASTRONOMY; AND AUTHOR OF
THE KEY TO HUTTON'S COURSE OF
MATHEMATICS.

Situation.

THE Mansion-House stands on Highgate Hill, on a spot the most healthy and picturesque in Middlesex, commanding a beautiful view of London and the Thames, with the hills of Surrey and Kent beyond them, the groves of Essex and Epping Forest on the left, and the village of Hampstead, almost contiguous, on the right; whilst a delightful assemblage of verdant fields, neat cottages, and shady lawns, constitute the fore ground of an enchanting prospect of more than 200 square miles.

Annexed to the Mansion-House (formerly the residence of Lady Cave) are eighteen acres of good garden, arable, and meadow land, affording Mr. Dowling a constant supply of milk, fruit, and every useful vegetable in season.

Stages and Post.

There are stage-coaches to and from Highgate every hour of the day, with three deliveries of letters and two departures of post.

Age, Accommodation, and Treatment.

Young Gentlemen, of all ages, are received and educated according to their future destination in life. The school-room and play-ground are dry, comfortable, and extensive, with arbours to retire to when it rains.

Mr. DOWLING treats his scholars with the utmost mildness, endeavours to form in them the habits and disposition of gentlemen, and cares for their health, comfort, and religious concerns with parental solicitude.

Each pupil has a separate bed, and the whole family breakfast and dine together.

Instruction.

THE ENGLISH LANGUAGE, ELEGANCE IN COMPOSITION, HISTORY, PLAIN AND ORNAMENTAL WRITING, AND ARITHMETIC, are taught indiscriminately to all the pupils, and on a plan which insures incredible success, in comparatively a short time.

The LATIN and GREEK languages, essential embellishments to a good education, are likewise indefatigably pursued, unless the age and views of the student determine otherwise.

The STUDY OF THE MATHEMATICS, not so much, indeed, for the exquisite pleasure wherewith it fills the mind in tracing the beautiful relations of quantity and figure, as for its communication of an invaluable habit of patient research, a love of truth, and a system of accurate reasoning, forms a grand feature in the plan of education.

And whether the youth be intended for the civil or military department of government, the land or sea service of his country, the Honourable East India Company's service, or the useful walks of private life, he is shortly made master of the branches more immediately requisite, by proper books, a well directed attention, and the use of the best Instruments.

The FRENCH LANGUAGE is also unremittingly followed up, and soon acquired in perfection, because, a considerable part of the family being French, that language is spoken in its purity in the House.

The Pupils have likewise an excellent opportunity of learning SPANISH and the other mercantile tongues.

BOOK-KEEPING, with all the routine of the Counting-House, and the forms of actual business, engages a reasonable portion of the time of the future Merchant; and it has been often found, that the information obtained in this department, under Mr. DOWLING, has been superior to that received by the generality of articled clerks.

Lectures.

To Mr. DOWLING's Lectures in Natural Philosophy and Astronomy, illustrated by the most complete set of apparatus possible, and one of the largest telescopes in England, the pupils have free access, and are assisted in their experiments and observations. Or, if the study of Chemistry be deemed expedient, there is a laboratory fitted up with all the necessary utensils, tests, and re-agents.

Dancing, Drawing, Music, and Fencing.

Approved Masters in each of these polite branches of Education, attend two or three times a week, at suitable hours, appropriated expressly for that purpose, without encroaching too much either on *Business* or *Recreation.*

Terms.

Pupils, under 10 years of age, pay 40 guineas per annum.
from 10 to 15 years, ... 50 guineas per annum.
from 15 to 20 years, ... 60 guineas per annum.

Subject to an extra charge for washing, and every branch of education not generally considered plain.

Vacations.

The usual vacations are, a month at Christmas, and at Midsummer; for either of which, if a pupil continues in the House, the charge is 5 guineas.

Mansion-House, Highgate,
4th Nov. 1817.

J. M'Creery, Printer,
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